MATHEMATICS AT ADVANCED LEVEL

(OPTIONAL)

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INDEX

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Chapter 1

Sets

1.1 Introduction

Let's begin by taking some examples where we use the word "set" without having formally studied it. You must have heard about different blood types. The primary system used for blood typing is the ABO system. The four major blood types under this system are:

- A
- B
- AB
- O

Blood types have very important applications. If a person of blood type 'A' needs blood transfusion, then which **set of people** can donate blood?

OR

A teacher recommends a **set of books** in algebra to the student.

1.2 Set

So, we may talk about a set of people or a set of books in a casual manner. Since set is often used in mathematics, we define this term first.

Definition of a Set

A set is a well-defined collection of objects. The objects in a set are the elements or members of the set. A collection is said to be *well-defined* if there is no confusion in deciding whether an object belongs to the collection or not.
"The three best students of a class" is not a set, because the term *best* may have different meanings for different people.
Some examples of set are:

(i) A set of students participating in a quiz contest.

(ii) A set of girls participating in Kho-Kho match.

On the other hand, collection of 3 most interesting books is not a set as different books may interest different students.

1.3 Representation of a Set

- **Roster Form:** One method of writing a set is to list all the elements of the set within braces separated by commas. For example, the set of vowels in English alphabet is written as V = $\{a, e, i, o, u\}$.

  Sets are denoted by capital letters.

This method is also known as tabular form.

The fact that "e is an element of the set V" is written as $e \in V$ where the symbol '$\in$' means "belongs to".

Similarly, we use the symbol $c \notin V$ to denote c is not an element of the set V.

Example 1: Write the following sets in Roster form

(i) Set of whole numbers less than or equal to 5.

(ii) Set of first 4 terms of the A.P., whose first term is –3 and common difference is 4.

Solution: (i) A = {0, 1, 2, 3, 4, 5} (ii) B = {–3, 1, 5, 9}

Can all sets be written in roster form? Think about it!

- **Set Builder Form**

Another method of writing a set is set builder form. In this method we specify the elements of the given set by its description. All the elements of the set possess a single common property which no element outside the set possess.

For example, the set V = $\{a, e, i, o, u\}$ can be written using this notation as

$$\{x \mid x \text{ is a vowel in English alphabet}\}$$

which is read as, "the set of all elements x such that x is a vowel of the English alphabet.

Example 2: Write the following sets in the set-builder form:

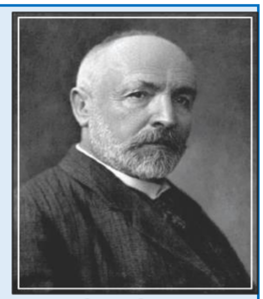
(i) $\{1, -1\}$ (ii) $\left\{\frac{2}{3}\right\}$ (iii) $\left\{\frac{1}{2}, \frac{1}{4}, 1, 2\right\}$

Solution: (i) $\{x \mid x \text{ is an integer and } x^2 = 1\}$

(ii) $\{x \mid x \text{ is fraction equivalent to } 0.\overline{6}\}$

(iii) $\left\{\frac{a}{b} \middle| a = 1 \text{ or } 4 \text{ and } b = 2 \text{ or } 4\right\}$

The theory of sets was developed by German mathematician Georg Cantor (1845-1918). He is considered the founder of set theory.


Georg Cantor (1845-1918)

Cantor developed interest in mathematics in his teens. He began his university studies in Zurich but shifted to Berlin the very next year in 1863, where he studied under the eminent mathematicians Karl Weierstrass, Ernst Kummer and Leopold Kronecker. He received his doctorate degree in 1867 in Number theory.

Cantor was also interested in philosophy and wrote papers relating his theory of sets to metaphysics. Cantor's contribution includes the discovery that the set of real numbers is uncountable. He is also noted for his contributions in analysis.

1.4 Finite and Infinite Sets

Can you make a list of students studying in your class? Of course it can be done easily, as the number of students are finite. Even the school's strength though, a large number, is finite. What about the set of natural numbers or set of real numbers between 2 and 3. Are they finite?

No these constitute infinite sets.

Some infinite sets are given below:

A = $\{x \mid x$ is a multiple of 5 and greater than 20$\}$

B = $\{x \mid x$ is a prime number$\}$

Example 3: State which of the following sets are finite or infinite:

(i) $\{x : x$ is an integer lying between 5 and 91$\}$

(ii) Set of coordinates of the points lying on a unit circle.

(iii) $\{x : x \in N$ and (x, y) lies on the line $2x + y = 8\}$

(iv) $\{x : x$ is the number of animals on earth$\}$

(v) $\{x : x$ is a digit in the decimal expansion of $\sqrt{2}\}$

Solution: (i) Since the set is $\{6, 7, 8, \ldots, 90\}$ so it is a finite set.

(ii) There are infinite points lying on a circle so its an infinite set.

(iii) Infinite points (x, y) where $x \in N$ lie on the line $2x + y = 8$. So set of values of abscissa forms an infinite set.

(iv) Set of animals on the earth is finite.

(v) $\sqrt{2}$ is an irrational number, so its decimal expansion is non-terminating, but the digits occurring in decimal expansion of $\sqrt{2}$ is finite.

Example 4: Write the following sets in roster form:

(i) $\{x \mid x$ is an odd natural number between 3 (excluding) and 11 (including)$\}$

(ii) $\{x \mid \sqrt{x}$ is a whole number less than or equal to 3$\}$

Solution: (i) $\{5, 7, 9, 11\}$ (ii) $\{0, 1, 4, 9\}$.

Example 5: Write the following sets in roster as well as set builder form.

(i) Real numbers between 2 and 5.

(ii) Fractions whose numerator and denominator are natural numbers and denominator exceeds the numerator by 1.

Is it possible to write in both the forms?

Solution: (i) Real numbers between 2 and 5.

Roster Form: Not possible to write the set in roster form, as we neither know its first element nor its last element. In fact, it's not possible to write two consecutive elements of this set.

Set Builder Form: $\{x : x \in R \text{ and } 2 < x < 5\}$

(ii) Fractions whose numerator and denominator are natural numbers and denominator exceeds the numerator by 1.

Roster Form: $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$

Set Builder Form: $\{x : x = \frac{n}{n+1}$, where $n \in$ natural number$\}$

Example 6: Use $\in$ or $\notin$ to indicate whether the given object is an element of the given set or not.

(i) $y \ldots \{a, x, y, z\}$

(ii) $4 \ldots \{1, 2, 5, 7, 11\}$

(iii) $2 \ldots \phi$

(iv) $5 \ldots \{x : x$ is a natural number $\leq 5\}$

Solution: (i) $\in$ (ii) $\notin$ (iii) $\notin$ (iv) $\in$

Example 7: Write each of the sets given below in the alternative form (roster or set builder)

(i) $\{2, 3, 5, 7, 11\}$

(ii) $\{1, 3, 5, 7, 9, 11\}$

Solution: (i) $\{x : x$ is a prime number less than $12\}$

(ii) $\{x : x$ is an odd natural number less than $13\}$

- **Is the order of elements in a set important?**

The elements of a set distinguish the set – not the order in which the elements are written.

Thus the three sets $\{1, 2, 3\}$, $\{1, 3, 2\}$ and $\{3, 2, 1\}$ are the same set.

- **Can a set have repeated elements?**

Think about it!

Try to form a set containing letters of the word "BANANA"

Is it {B, A, N, A, N, A}? If it is correct, how many elements does the set contains, 6 or 3.

In fact, repetition of elements is not allowed in a set. A set contains only distinct elements. So the set containing the letters of the word "BANANA" is {B, A, N}.

Let us define some more type of sets:

1.5 Empty or Null Set

Consider the set {x : x is a prime number which is composite}

Is there a number which is both prime as well as composite?

Since no prime is composite, so the above set can be written in roster form as { }.

We say a set which does not contain any element is called an empty, null or void set. An empty set is denoted by the symbol ϕ or { }.

Example 8: Which of the following are empty sets:

(i) Set of even prime numbers.

(ii) Set of composite numbers having atmost 2 factors.

(iii) Set of numbers which are both rational and irrational.

(iv) Set of irrational numbers whose decimal expansion terminates.

Solution: (i) {2}; It is not an empty set.

(ii) Since a composite number has more than two factors, so it is an empty set.

(iii) Since no rational number is irrational, so set of numbers which are both rational and irrational is an empty set.

(iv) The decimal expansion of irrational number is neither terminating nor recurring. So it is an empty set.

1.6 Equality of Sets

Let A and B be two sets. We say A = B, if A and B have the same elements.

If two sets A and B are not equal, we write A $\neq$ B.

Some example of equal sets are:

{2, 5, –1, 0} and {0, –1, 5, 2} ;

{1, 2, 2, 2, 7, 7, 7, 7, 7} and {1, 2, 7} ;

{x : x is a prime divisor of 6} and {x : x is a pair of consecutive numbers that are prime}

EXERCISE 1.1

1. List the elements of the following sets :

(a) {x : x is an integer and $x^2 = 9$}

(b) {x : x is a positive integer less than 5}

(c) {x : x is even natural number divisible by 5}

(d) {x : $x \in$ N and $x < -1$}

2. Determine which elements of the set

$$A = \left\{-5, -\sqrt{3}, -\frac{1}{2}, 0, \frac{2}{5}, \pi, 13.4, \frac{1}{3}, \frac{19}{2}\right\}, \text{ are}$$

(a) Natural numbers
(b) Whole numbers
(c) Integers
(d) Rational numbers
(e) Real numbers

3. Write the following sets in roster form:

(a) $\{x : x$ is a two digit number and the sum of the digits is 5$\}$

(b) $\{x : x$ is an integer and $|x| \leq 9\}$

(c) $\{x : x$ is letter of the word "SWEET"$\}$

(d) $\{x : x = \frac{n+1}{n}$, where n is a natural number and $n < 6\}$

(e) $\{x : x$ is a composite number$\}$

4. Write the following sets in set-builder form.

(i) $\{2, 4, 6, 8, \ldots\}$
(ii) $\{3, 6, 9, 12, 15\}$
(iii) $\{1, 4, 9, 16, \ldots\}$
(iv) $\{8, 9, 10, 11, \ldots\}$
(v) $\{1, 2, 3, 6\}$

Can two different sets have the same roster form?

5. Which of the following pairs of sets are equal.

(i) {D, E, C, E, N, T} and {C, E, N, T, D}

(ii) $\{a, b, \pi, \sqrt{2}\}$ and $\{a, \pi, \sqrt{2}, b\}$

(iii) $\{x : x$ is zero of the polynomial $x^2\}$ and $\{x : x$ is the root of the equation, $x^2 = 0\}$

(iv) $\{x : x$ has numerical value less than or equal to 1$\}$ and $\{x : x$ is the root of the equation, $x^2 - 1 = 0\}$

(v) $\{5, 10, 15, 20\}$ and $\{5, 10, 15, 20, \ldots\}$

(vi) ϕ and $\{\phi\}$

6. State which of the following sets are finite or infinite.

(i) $\{x : x \in Z$ and $(x - 1)(x + 2)(x - 3) = 0\}$

(ii) $\{x : x$ and 2 are coprime$\}$

(iii) $\{x : x$ is a rational number between 3 and 4$\}$

(iv) $\{x : x$ is an integer and $|x| \geq 5\}$

1.7 Subset

Consider the set X of students who live in the vicinity of 5 km radius around your school.

These students obviously along with others belongs to the set Y of all students in your class.

Since every student of set X belongs to or is contained in set Y, we say X is a subset of Y.

Definition: The set A is said to be a subset of B if and only if every element of A is also an element of B.

Symbolically, we write it as $A \subseteq B$.

If A is not a subset of set B, we write it as $A \not\subseteq B$.

We say $A \subseteq B$ if $a \in A \Rightarrow a \in B$.

Note that if A is a subset of B and $A \neq B$, then we say A is a proper subset of B and denote it by $A \subset B$

- Is an empty set ϕ, a subset of a set P containing at least one element?

Yes, the empty set ∅ is a subset of set P. In fact, the empty set is a subset of **every** set.

Example 9: Use the notation $\subseteq$ to denote which set is the subset of the other in the following problems.

(i)	$A = \{2, 3, p\}$	$B = \{1, 2, 3, p, q\}$
(ii)	$A = \{2, 3, 5, 7\}$	$B = \phi$
(iii)	$A = \{3, 8, 9, 0\}$	$B = \{0, 9, 8, 3\}$
(iv)	$A = \{x \mid x = 2n, \text{ where } n \in N]$	$B = \{x \mid x = 4n, \text{ where } n \in N\}$

Solution:

(i) $A \subseteq B$

(ii) Since ϕ is subset of every set, So $B \subseteq A$

(iii) Since A = B, therefore each is a subset of the other i.e., $A \subseteq B$ as well as $B \subseteq A$.

(iv) $A = \{2, 4, 6, 8, 10, 12, \ldots\}$

$B = \{4, 8, 12, 16, 20, \ldots\}$

So $B \subseteq A$.

1.8 Cardinality of a Set

Let A be any set. If there are exactly m distinct elements in A, we say, cardinality of set A is m. Symbolically we write it as

$$n(A) = m, \text{ where } m \text{ is a non-negative integer,}$$

For example, if $A = \{1, 2, 3, \ldots, 9\}$ then $n(A) = 9$ and if $B = \left\{\pi, \sqrt{3}, 7\frac{1}{2}, 0\right\}$ then $n(B) = 4$.

1.9 Power Set

Consider the set D = $\{a, b\}$. Can you write all its subsets?

Are $\{a\}$, $\{b\}$ its only subsets?

A little thinking would suggest some are still left.

It has two more subsets i.e. $\{a, b\}$ and ϕ.

So the set has 4 subsets in all i.e., $\{a\}$, $\{b\}$, $\{a, b\}$, ϕ.

If these subsets are written in a set form, we call it a power set of D.

$$\text{i.e., } P(D) = \{\{a\}, \{b\}, \{a, b\}, \phi\}$$

Note:
- $\{a\}$ and $\{a, b\}$ are elements of the set P(D).
- Neither a, nor a, b taken together are the elements of the set P(D).

Example 10: What is the power set of A = {0, 1, 2}?

Solution: Since the power set of A is again a set containing all its subsets, so

$P(A) = \{\phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

Symbolically, we denote it by P(A).

Example 11: What is the power set of an empty set? What is the power set of $\{\phi\}$?

Solution: Since every set is a subset of itself. So ϕ is a subset of ϕ. Hence $P(\phi) = \{\phi\}$.

Also the set $\{\phi\}$ has exactly two subsets ϕ and $\{\phi\}$ itself hence $P(\{\phi\}) = \{\phi, \{\phi\}\}$.

A set containing n elements has 2^n subsets. If $n(A) = p$, where p is a whole number then $n[P(A)] = 2^p$.

Consider the set A = {1, 2} then it has 2^2 i.e., 4 subsets and $2^2 - 1 = 3$ proper subsets (i.e. ϕ, {1} and {2}).

1.10 Universal Set

In sets, the elements that we consider are usually limited to a specific all-encompassing set. For example, when we take sets of students belonging to a class or different sections of the same class or students of an editorial team, they all study in the same school. So the universal set denoted by 'U' is the set of all students of the school.

Similarly, if we discuss set of natural numbers, rational numbers and irrational numbers, then real numbers is the appropriate universal set. Universal set is denoted by U.

EXERCISE 1.2

1. Fill in the blanks with symbol $\subset$ or $\not\subset$.

(i) $\{2, 3, 4\}$ .... $\{1, 2, 3, 4, 5\}$

(ii) $\{x \mid x$ are triangles in a plane$\}$ .... $\{x \mid x$ are polygons in a plane$\}$

(iii) $\{x$ is an integer$\}$ .... $\{x : x$ is a multiple of $4\}$

(iv) ϕ .... $\{\phi\}$

(v) $\{x \mid x = \frac{m-1}{m}$, where m is a non-zero integer$\}$ .... $\{x \mid x$ is a rational number$\}$

(vi) $\{x \mid x = n^2\}$ .... $\{x \mid x = n^3\}$, (where n is a natural number)

(vii) $\{x \mid x \in R\}$ .... $\{x \mid x = 2^n\}$, (where R is a real number)

2. Determine whether the following statements are true or false.

(i) $1 \in \{1\}$

(ii) $\{2\} \in \{2\}$

(iii) $\{2\} \in \{\{2\}\}$

(iv) $\phi \in \{1, 2, 3\}$

(v) $\phi \subseteq \{1, 2, 3\}$

3. Write the power set of the following sets:

(i) $\{1\}$

(ii) $\{p, q\}$

(iii) $\{1, 2, 5\}$

(iv) $\{\phi, \{\phi\}\}$

4. What is the cardinality of the following sets:

(i) $\{a\}$

(ii) $\{a, \{a\}\}$

(iii) $\{\phi, 1, 2, \{1, 2\}\}$

(iv) $\{1, \{1\}, \{1, \{1\}\}\}$

(v) $\{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$

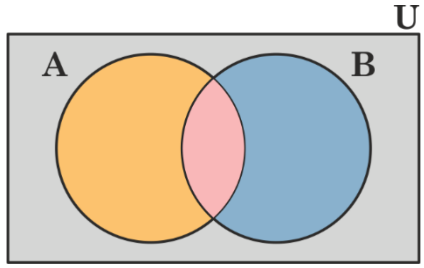
5. Let A be a set and $n(A) = 10$, then find the value of $n[P(A)]$? What if A has 100 elements?

1.11 Venn Diagram

A Venn diagram is a visual or pictorial representation of sets. This representation is known as Venn diagram after the English mathematician and philosopher John Venn (1834-1923). He studied in Cambridge, London. These diagrams are used to illustrate relationships between sets and draw logical results.

In the following figure two sets are represented by two circles enclosed in a rectangle.

The rectangle represents the universal set U. The circles may be overlapping or distinct depending upon they have common elements or not.



1.12 Set Operations

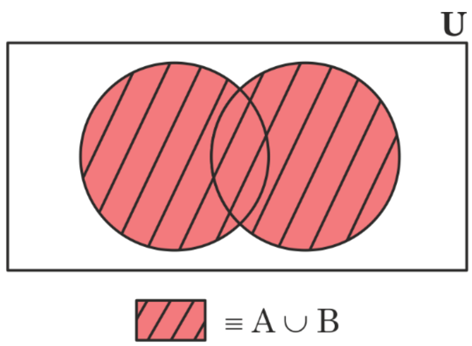
Two sets can be combined in many different ways. For instance, if we have set of students who play cricket and a set of students who play soccer, then we can form a set of students who play either cricket or soccer. We may also consider students who play both cricket and soccer. This is possible by performing certain operations on two sets to give another set.

Let us study these operations in detail.

1.12.1 Union of Two Sets

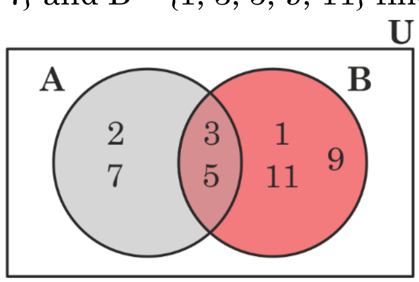
Let A and B be any two sets. Then union of A and B is the set containing elements of A or B or both A and B.

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$



Note: '$A \cup B$' is also denoted by A or B.

Example 12: If $A = \{2, 3, 5, 7\}$ and $B = \{1, 3, 5, 9, 11\}$ find $A \cup B$.



Solution: Since $A \cup B$ consists of all the elements of A as well as B.

Hence, $A \cup B = \{1, 2, 3, 5, 7, 9, 11\}$

Example 13: Find the union of the following pair of sets.

(i) A = {2, 5, 9}, B = {1, 4, 7}

(ii) C = {3, 5, 6}, D = {3, 4, 5, 6, 9}

(iii) P = {a, b, d, e}, Q = {b, c, e, f, g}

Solution: (i) $A \cup B$ = {1, 2, 4, 5, 7, 9}

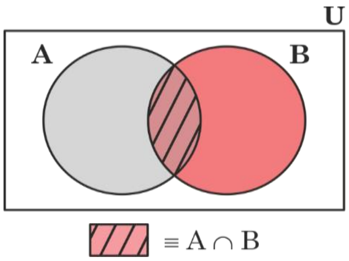
(ii) $C \cup D$ = {3, 4, 5, 6, 9}

(iii) $P \cup Q$ = {a, b, c, d, e, f, g}

1.12.2 Intersection of Two Sets

Let A and B be two sets. Then the intersection of A and B is the set that contains elements present in both A and B.

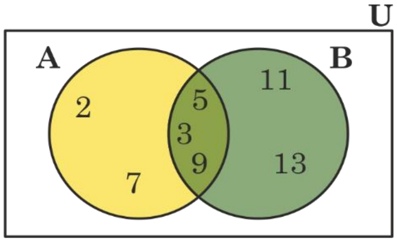
Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$



Note: '$A \cap B$' is also denoted by 'A and B'.

Example 14: Find the intersection of the following two sets

A= {2, 3, 5, 7, 9} B= {3, 5, 9, 11, 13}



Solution: Since $A \cap B$ consists of elements that are common to A and B.

Hence, $A \cap B$ = {3, 5, 9}.

Example 15: Find the intersection of the following pairs of sets

(i) {a, b, f} and {d, e, f, g}

(ii) {1, 2, 3, 6, 9} and {1, 4, 5, 9, 13}

(iii) ϕ and {c, d, e}

(iv) {x : x is a natural number greater than 4 and less than 10} and {x : x is a factor of 12}

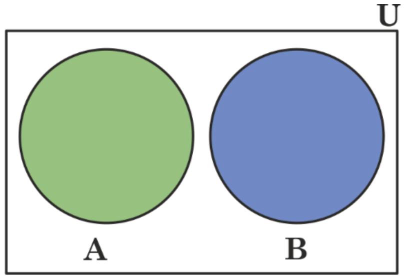
Solution: (i) {f} (ii) {1, 9} (iii) ϕ (iv) {6}

1.12.3 Disjoint Sets

Two sets A and B are disjoint if they have no element in common.

Therefore, the intersection of two disjoint sets is an empty set

$$A \cap B = \phi$$



Disjoint Sets

Example 16: Identify pair(s) of disjoint sets from the following:

$A = \{1, 3, 4\}$

$B = \{5, 6, 7, 8, \ldots\}$

$C = \{x : x \text{ is a prime factor of } 36\}$

$D = \{5, 7, 11, 13\}$

Solution: Since, $A \cap B = \phi$; A, B are disjoint

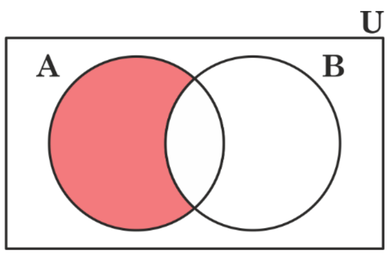
$A \cap D = \phi$; A, D are disjoint

$C \cap D = \phi$; C, D are disjoint.

1.12.4 Difference of Sets

Let A and B be two sets. The difference of A and B, denoted by $A - B$ is the set containing elements which are in A but not in B.

Symbolically, $A - B = \{x : x \in A \text{ and } x \notin B\}$



 $\equiv A - B$

The difference of $\{2, 5, 7\}$ and $\{1, 2, 4\}$ is $\{5, 7\}$.

Example 17: Find the difference of A and B from the following pair of sets.

(i) $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 6, 8\}$

(ii) $A = \{a, b, p, q\}$, $B = \{b, q\}$

(iii) $A = \{1, 5, 9\}$, $B = \{1, 2, 4, 5, 7, 9\}$

Solution: (i) $A - B = \{1, 3, 5, 7, 9\}$

(ii) $A - B = \{a, p\}$

(iii) $A - B = \phi$

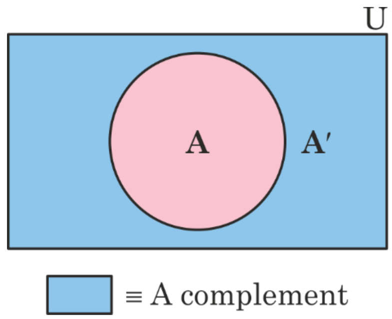
1.13 Complement of a Set

Let us learn complement of a set by taking a simple example.

Consider the students of your class as universal set. Let A be the set of boys in the class and B be the set of girls in the class. Then complement of boys is the set of students other than the boys i.e. the girls. Let us give the formal definition of the complement.

Definition of Complement

The complement of set A denoted by A′ is defined by $A' = \{x : x \in U \text{ and } x \notin A\}$



Example 18: Given,

$U = \{x \in N : x \leq 9\}$

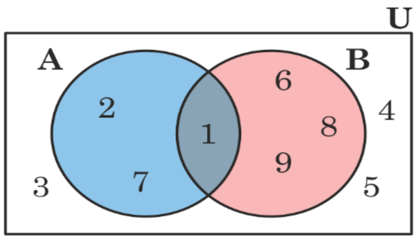
$A = \{1, 2, 7\}$

$B = \{1, 6, 8, 9\}$

find the following:

(a) A' (b) B' (c) $(A \cup B)'$ (d) $A' \cap B'$

Solution: $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$



(a) $A' = \{3, 4, 5, 6, 8, 9\}$

(b) $B' = \{2, 3, 4, 5, 7\}$

(c) $(A \cup B)' = \{1, 2, 6, 7, 8, 9\}' = \{3, 4, 5\}$

(d) $A' \cap B' = \{3, 4, 5, 6, 8, 9\} \cap \{2, 3, 4, 5, 7\} = \{3, 4, 5\}$

Note that $(A \cup B)' = A' \cap B'$. This law is known as **De-Morgan's Law**.

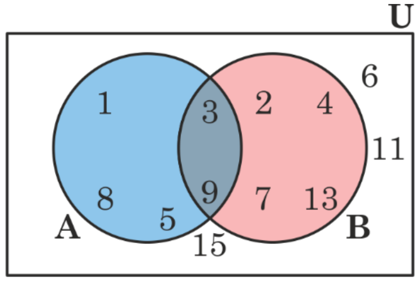
Example 19: Given A = {1, 3, 5, 8, 9}, B = {2, 3, 4, 7, 9, 13}

and U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15}

Use Venn diagram to verify the following De-Morgan's Laws.

(i) $(A \cap B)' = A' \cup B'$ (ii) $(A \cup B)' = A' \cap B'$

Solution: Representing given sets by Venn diagram we get



(i) $(A \cap B)' = U - (A \cap B) = \{1, 2, 4, 5, 6, 7, 8, 11, 13, 15\}$

$A' \cup B' = \{2, 4, 6, 7, 11, 13, 15\} \cup \{1, 5, 6, 8, 11, 15\}$

$= \{1, 2, 4, 5, 6, 7, 8, 11, 13, 15\}$

Hence, $(A \cap B)' = A' \cup B'$

(ii) $(A \cup B)' = U - (A \cup B) = \{6, 11, 15\}$

$A' \cap B' = \{2, 4, 6, 7, 11, 13, 15\} \cap \{1, 5, 6, 8, 11, 15\}$

$= \{6, 11, 15\}$

Hence, $(A \cup B)' = A' \cap B'$

1.14 Application of Sets

The concept of cardinal number finds many practical applications in real life. The theory of sets and the operations on them, provides some very useful formulae.

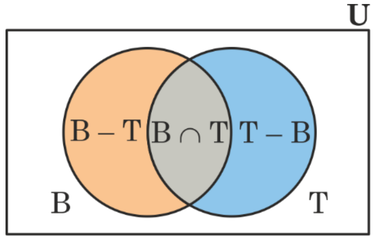
Let us now discuss and enlist a few observations which can be very easily verified using Venn diagrams.

1.14.1 If A and B are two finite sets, then their cardinal numbers are related as below:

1. n(Either in A or in B) = $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
2. n(Only in A, not in B) = $n(A - B) = n(A) - n(A \cap B)$
3. n(Neither in A nor in B) = $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$
4. n(Only in one of them) = $n[(A - B) \cup (B - A)] = n(A) + n(B) - 2\,n(A \cap B)$

Example 20: In a class with 40 students, 22 play badminton, 11 play both badminton and table tennis and 16 play neither badminton nor table tennis. How many play table tennis but not badminton?

Solution: Let B represents the set of students who play badminton and T represents the set of students who play table tennis.



We are given, $n(U) = 40$, $n(B) = 22$, $n(B \cap T) = 11$, $n(B \cup T)' = 16$

$$n(B \cup T) = n(U) - n(B \cup T)' = 40 - 16 = 24$$

Now, $n(B \cup T) = n(B) + n(T) - n(B \cap T)$

i.e., $24 = 22 + n(T) - 11$

or $24 = 11 + n(T)$

$n(T) = 13$ i.e. 13 students play table tennis

n(Play table tennis but not badminton)

$= n(T) - n(B \cap T) = 13 - 11 = 2$ students

2 students play table tennis but not badminton.

Example 21: Each student from a group of 120 university students participated in teaching either a language or mathematics to the needy students. It is found that 92 of them can teach a language to the needy students and 46 can teach mathematics.

(a) Find the number of students who can teach both the subjects.

(b) Find the number of students who can teach only one of the two subjects.

Solution: If L denotes the set of students who can teach a language, M denotes the set of students who can teach mathematics

$n(L) = 92$, $n(M) = 46$ and $n(L \cup M) = 120$

Applying the relation

(a) $n(\text{Either in L or in M}) = n(L \cup M)$

$= n(L) + n(M) - n(L \cap M)$ we get

$n(L \cap M) = n(L) + n(M) - n(L \cup M)$

$= 92 + 46 - 120 = 18$

Therefore 18 students can teach both the subjects.

(b) $n(\text{Exactly in one of them}) = n[(L - M) \cup (M - L)]$

$= n(L) + n(M) - 2n(L \cap M)$

$= 92 + 46 - 2 \times 18$

$= 138 - 36$

$= 102$

Therefore 102 students can teach only one of the two subjects.

Note: You can also use the Venn diagram to solve this example.

Example 22: In a survey of 100 students regarding their preference for two subjects—Physics (P) and Chemistry (C), it is observed that 70 preferred Physics and 60 preferred Chemistry.

(i) Find the maximum possible number of students who neither like Physics nor Chemistry.

(ii) Find the minimum possible number of students who like both Physics and Chemistry.

Solution: We know that, $n(P \cup C) = n(P) + n(C) - n(P \cap C)$

(i) For maximum number of students who like neither of the subjects, we need $(P \cup C)'$

$n(P \cup C)' = n(U) - n(P \cup C)$ will be maximum if $n(P \cup C)$ is minimum. This happens when one of the two sets is a subset of the other, so that $P \cap C$ becomes the maximum.

The maximum possible value of $n(P \cap C)$ is the cardinal number of the smaller set, which is 60. (Means if everyone who likes Chemistry also likes Physics).

Thus, Minimum $n(P \cup C) = 70 + 60 - 60 = 70$.

Therefore, maximum number of students who neither like Physics nor Chemistry = 100 – 70 = 30

(ii) We know that $n(P \cup C) \leq n(U)$

This means $70 + 60 - n(P \cap C) \leq 100$

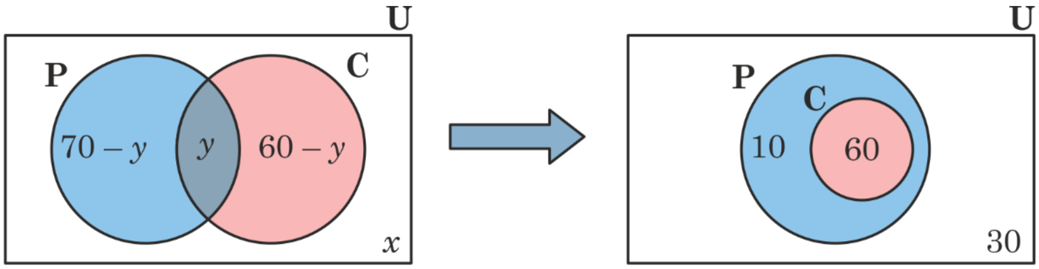
Therefore, $n(P \cap C) \geq 30$

Thus the minimum possible number of students who like both Physics and Chemistry is 30.

Alternate solution:

Let x be the number of students who like neither Physics nor Chemistry. These students sit outside the two circles but inside the rectangular Universal set (U).

So on the basis of information the following Venn diagram is drawn.



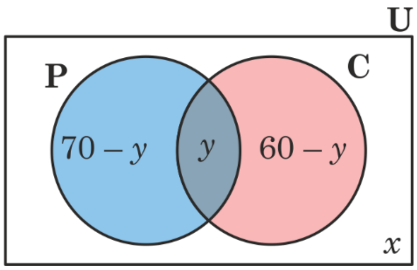
(i) Thus, to make x as large as possible, we must make the "space" occupied by the circles $(P \cup C)$ as small as possible. To make the union of two circles small, we must overlap them as much as possible. The most they can possibly overlap is the size of the smaller set. So, we put all 60 Chemistry students inside the Physics circle.

Therefore, Maximum value of neither Physics nor Chemistry = 30

(ii) Since $n(P \cup C) = n(P) + n(C) - n(P \cap C)$

$n(P \cap C) = n(P) + n(C) - n(P \cup C)$

Now $n(P \cap C)$ will be minimum if $n(P \cup C)$ is maximum.



The maximum possible value of $n(P \cup C) = 100$ (as there are a total of 100 students).

Hence, minimum value of $n(P \cap C)$

$= n(P) + n(C) - n(P \cup C)$

$= 70 + 60 - 100 = 30$

1.14.2 If A, B and C are three finite sets, then the relation between the cardinal numbers is given below:

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - [n(A \cap B) + n(B \cap C) + n(C \cap A)] + n(A \cap B \cap C)$

Example 23: A market research group conducted an online survey of 1000 consumers and found that 35% of the consumers had rated a shampoo of type A by 5-stars. While 30% of the consumers rated the shampoo of type B by 5-stars and 250 consumers rated the shampoo of type C by 5-stars. It was observed that 200 consumers gave 5-star rating to the shampoos of both the types A and B, 150 consumers gave it to the shampoos of both the types B and C and 15% gave a 5-star to both A and C and 100 consumers rated all of them with 5-star.

(i) Find the percentage of consumers who gave 5-star rating to only one type of shampoo.

(ii) Find the number of consumers who did not give 5-star rating to any of the shampoos.

Solution: Let A, B and C represent the sets of consumers who rated 5-stars to the shampoos A, B and C respectively. We observe

$n(U) = 1000$; $n(A) = 35\%$ of $1000 = 350$;

$n(B) = 30\%$ of $1000 = 300$ and $n(C) = 250$

$n(A \cap B) = 200$; $n(B \cap C) = 150$;

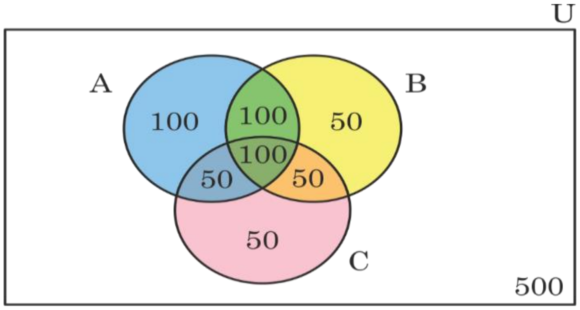
$n(A \cap C) = 15\%$ of $1000 = 150$ and $n(A \cap B \cap C) = 100$

Using Venn diagram we find:

$n(\text{only in A and B}) = n(A \cap B) - n(A \cap B \cap C) = 200 - 100 = 100$

$n(\text{only in B and C}) = n(B \cap C) - n(A \cap B \cap C) = 150 - 100 = 50$

$n(\text{only in A and C}) = n(A \cap C) - n(A \cap B \cap C) = 150 - 100 = 50$



(i) Thus, from Venn diagram, Number of consumers who gave 5-star rating to only one type of shampoo = 100 + 50 + 50 = 200

So, percentage of consumers who gave 5-star rating to only one type of shampoo $= \frac{200}{1000} \times 100 = 20\%$

(ii) Now, number of consumers who did not give 5-star rating to any of the shampoos $n(U) - n(A \cup B \cup C) = 1000 - 500 = 500$

EXERCISE 1.3

1. Find the union of sets A and B i.e. $A \cup B$, in each of the following pairs.

(i) $A = \{1, 2, 3, 7\}$, $B = \{2, 7, 9\}$

(ii) $A = \{a, b, d, e\}$, $B = \{a, e, i, o, u\}$

(iii) $A = \{x : x \text{ is natural number} > 5\}$, $B = \{x : x \text{ is natural number} < 5\}$

(iv) $A = \phi$, $B = \{2, \sqrt{2}, -1, 0\}$

2. Evaluate each of the following.

(i) $\{1, 2\} \cap \{1, 2, 5\}$

(ii) $\{1, 3, 5, 7, 9\} \cap \{2, 4, 6, 8\}$

(iii) $\{g, o, a, t\} \cap \{c, a, t\}$

(iv) $\{x : x \text{ is an integer}\} \cap \{x : x \text{ is a negative integer}\}$

3. Which of the following sets are disjoint?

(i) $\{x : x \text{ is a multiple of } 2\}$ and $\{x : x \text{ is a multiple of } 3\}$

(ii) $\{e, \pi, \sqrt{2}, 0\}$ and $\left\{e^2, \frac{\pi}{2}, \sqrt{3}, 1\right\}$

(iii) $\{x : x \text{ is a real number}\}$ and $\{x : x \text{ is an irrational number}\}$

4. Find A – B in each of the following.

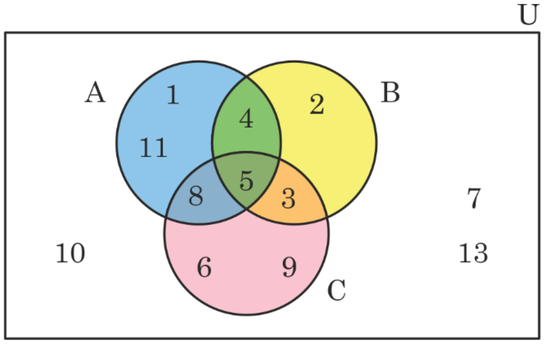
(i) $A = \{1, 3, 5, 8\}$, $B = \{3, 7, 8, 9\}$

(ii) $A = \{3, 0, 8\}$, $B = \{1, 3, 0, 8, 9\}$

(iii) $A = \{2, 6\}$, $B = \{1, 3, 5, 9\}$

5. Use the Venn diagram given below to answer the questions that follow.

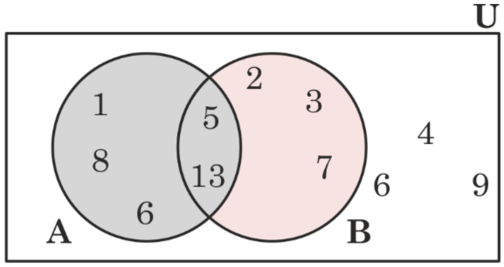
Hint: You can find sets A, B, C and universal set U from the given Venn diagram.



(i) A' (ii) B' (iii) $(A \cap B)'$

(iv) $A' \cup B'$ (v) $A \cap B \cap C$ (vi) $A \cap (B \cup C)$

6. Verify $A - B = A \cap B'$ using the Venn diagram given below:



7. For a competitive exam, 85% of students opted for a Mock Test in Mathematics and 75% opted for a Mock Test in Science.

(a) What is the minimum possible percentage of students who opted for both tests?

(b) If 10% opted for neither, how does the minimum percentage for both change?

8. Let S be a set of 50 people. 35 people speak English and 25 speak Hindi. If k be the number of people who speak only English, then find the possible value of k, assuming every person speaks at least one of the two languages.

9. An organization awarded certificates to its 56 students for at least one of the three activities of Origami, Instrumental music and Fine arts. If 17 students received the certificates for Origami, 28 for Instrumental music, 25 for Fine arts and only 4 students got the certificates for all the three activities. Find the number of students who received the certificates for exactly two activities.

10. A survey of a group of 100 students in an international school revealed that 60 students could speak English, 50 students could speak German and 35 students could speak Spanish. Further 40 students could speak both English and German, 30 could speak both German and Spanish, 25 could speak both English and Spanish and 25 could speak all the three languages. Let E represent the set of students who speak English, G represents the set of students who speak German and S represent the set of students who speak Spanish. Answer the following using Venn diagram.

(a) How many students could speak at least two languages?

(b) How many students could speak at most one language?

(c) How many students could not speak any of the three languages?

Summary

(1) A set is a well-defined collection of objects.

(2) A set is represented in two forms
- Roster Form
- Set-Builder Form

(3) A set is finite if its number of elements is a natural number. Else it is infinite.

(4) A is subset of B if every element of A is contained in B. Symbolically it is denoted by $A \subseteq B$.

(5) **Cardinality** of a set is the number of distinct elements in it. It is denoted by $n(A)$.

(6) The set of all subsets of a set A is called **power set** of A and is denoted by P(A).

(7) **Operation of sets.**
- **Union** of sets A and B is a set containing elements of A or elements of B or both. It is denoted by $A \cup B$.
- **Intersection** of sets A and B is a set containing elements of A and B. It is denoted by $A \cap B$.

(8) **Disjoint sets**

Two sets A and B are said to be disjoint if their intersection is an empty set

$$\text{i.e., } A \cap B = \phi$$

(9) **Difference** of two sets A and B is a set containing elements of A which are not in B

$$\text{i.e., } A - B = \{x : x \in A \text{ and } x \notin B\}$$

(10) **Complement** of a set A with respect to universal set U is a set containing the elements of U which are outside A

$$\text{i.e., } A' = U - A.$$

(11) **Applications of Sets** using **cardinal relations and Venn diagrams**

11.1 If A and B are two finite sets, then their cardinal numbers are related as below:
- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$
- $n((A - B) \cup (B - A)) = n(A) + n(B) - 2n(A \cap B)$

11.2 If A, B and C are three finite sets, then their cardinal numbers are related as follows:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - [n(A \cap B) + n(B \cap C) + n(C \cap A)] + n(A \cap B \cap C)$$

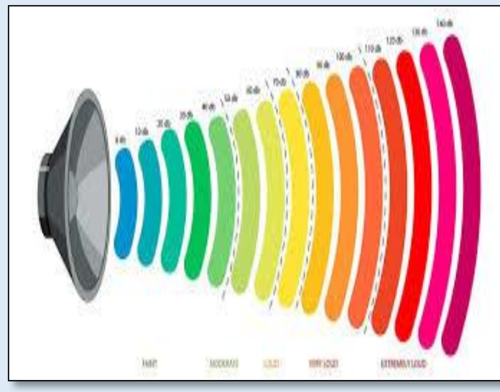
Chapter 2

Logarithms

2.1 Introduction to Logarithms

Imagine a world before calculators and computers when mathematicians had to do complex calculations involving multiplication and division of large numbers. It took tremendous time and effort often involving lengthy calculations.

Opening Puzzle: The Sound of Numbers



In a music studio, the sound engineer says:

"This speaker produces a sound that is 1000 times more intense than the softest sound we can hear."

Instead of saying "1000 times," scientists say:

Sound Level = $\log_{10} 1000 = 3$

Why 3?

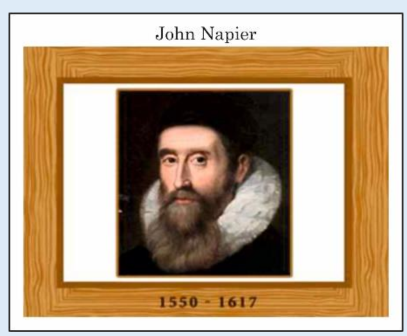
Because: $10^3 = 1000$

Then came **John Napier** with a revolutionary idea of effectively turning "multiplication into addition," and "division into subtraction" using logarithms. Logarithms is a tool that helps to do large calculations easily.

This approach saved scientists and fellow mathematicians a lot of tedious calculations.

Logarithms were introduced by John Napier (1550–1617), a Scottish mathematician. His method was different from the modern approach and was based on the relationship between arithmetic and geometric sequences. Later, Henry Briggs refined this idea and developed common logarithms (base 10), which made calculations easier.

Logarithm tables were widely used in science and engineering to simplify multiplication and division until electronic calculators became common. Even today, logarithms remain important in mathematics, especially natural logarithms (base *e*), which are widely used in calculus.



2.2 Understanding Logarithms as the Inverse of Exponents

We have earlier learnt about squares and cubes. For example:

- $10^2 = 100$ (10 squared is 100)
- $10^3 = 1000$ (10 cubed is 1000)

In these cases, we have taken the **base** 10, their **exponents** as 2 and 3. We find the value of base raised to the given exponent. But what if we know the resultant value and the base, and we want to find the exponent?

This power or exponent to which the base is raised is called a **Logarithm**.

- **Exponential Form:** $10^x = 100$
- **Logarithmic Form:** $\log_{10} 100 = x$

We already know about **exponents**: $2^3 = 8$

This means:

- Base = 2
- Exponent = 3
- Resultant = 8

Let us ask a different question: **2 raised to what power gives 8?**

That power is 3. So we say that logarithm of 8 to the base 2 is 3.

2.2.1 Understanding Logarithms through powers of 10

Let us look at powers of 10:

Powers of 10	Expressed in logarithmic form:
$10^0 = 1$ $10^1 = 10$ $10^2 = 100$ $10^3 = 1000$ $10^4 = 10000$	$\log_{10} 1 = 0$ $\log_{10} 10 = 1$ $\log_{10} 100 = 2$ $\log_{10} 1000 = 3$ $\log_{10} 10000 = 4$
$10^{-4} = 0.0001$ $10^{-3} = 0.001$ $10^{-2} = 0.01$ $10^{-1} = 0.1$ $10^0 = 1$	$\log_{10} 0.0001 = -4$ $\log_{10} 0.001 = -3$ $\log_{10} 0.01 = -2$ $\log_{10} 0.1 = -1$ $\log_{10} 1 = 0$

Definition: For any positive number b (where $b > 0$ and $b \neq 1$) and a positive number a if $b^x = a$ then $\log_b a = x$. We read this as " logarithm of a to the base b is x" .

For example, $32 = 2^5$ we can write $5 = \log_2 32$. These two statements are **equivalent** and we indicate this by using the symbol $\Leftrightarrow$. We write: $32 = 2^5 \Leftrightarrow \log_2 32 = 5$.

In general, $b^x = a$ where $a, b > 0$ and $b \neq 1$ and $\log_b a = x$ are **equivalent** statements and we write

$$b^x = a \quad \Leftrightarrow \quad \log_b a = x$$

Example 1: Write an equivalent form:

(a) Logarithmic form for $9^{\frac{1}{2}} = 3$

(b) Exponential form of $\log_5 625 = 4$

> **Math Talk:**
> Can we find logarithm of a negative number? Why or why not?

Solution:

(a) $9^{\frac{1}{2}} = 3 \quad \Leftrightarrow \quad \log_9 3 = \frac{1}{2}$

(b) $\log_5 625 = 4 \quad \Leftrightarrow \quad 5^4 = 625$

> **Math Talk:** Is this true for all bases?

Example 2: Rewrite $2^0 = 1$ in logarithmic form:

Solution: $2^0 = 1 \quad \Leftrightarrow \quad \log_2 1 = 0$

EXERCISE 2.1

1. Write an equivalent logarithmic statement for:

(a) $5^3 = 125$

(b) $(2)^5 = 32$

(c) $(7)^{-1} = \frac{1}{7}$

(d) $(3)^{\frac{-1}{2}} = \frac{1}{\sqrt{3}}$

2. Write an equivalent exponential statement for:

(a) $\log_2 16 = 4$

(b) $\log_9 81 = 2$

(c) $\log_5 \sqrt{5} = \frac{1}{2}$

(d) $\log_2 \left(\frac{1}{2}\right) = -1$

3. Find the value of

(a) $\log_{10} 1000$

(b) $\log_6 36$

(c) $\log_2 64$

2.3 Logarithms Properties

The definition and the assumptions of logarithm are as follows:

Definition of Logarithm:

For $a > 0$, $a \neq 1$ and $x > 0$,

If $y = \log_a x$, then $a^y = x$.

Equivalently, $\log_a x = y \iff a^y = x$.

Assumptions:

- $a > 0, a \neq 1$ (The Base)
- $M > 0, N > 0$ (Positive real number)

These conditions ensure all logarithms in the following proofs, are defined.

Properties of Logarithms

Product Rule

Statement: $\log_a(MN) = \log_a M + \log_a N$

Proof: Let $x = \log_a(MN)$, $y = \log_a M$, $z = \log_a N$

Then, $a^x = MN$, $a^y = M$ and $a^z = N$

Therefore, $a^{y+z} = a^y a^z = MN \Rightarrow y + z = \log_a(MN)$

Thus, $\log_a M + \log_a N = \log_a(MN)$

Quotient Rule

Statement: $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

Proof: Let $x = \log_a\left(\frac{M}{N}\right)$, $y = \log_a M$, $z = \log_a N$

Then, $a^x = \left(\frac{M}{N}\right)$, $a^y = M$ and $a^z = N$

Therefore, $a^{y-z} = \frac{a^y}{a^z} = \left(\frac{M}{N}\right) \Rightarrow y - z = \log_a\left(\frac{M}{N}\right)$

Thus, $\log_a M - \log_a N = \log_a\left(\frac{M}{N}\right)$

Power Rule

Statement: $\log_a(M^k) = k\log_a M$

Proof: Let $x = \log_a(M^k)$ and $y = \log_a M$

Then, $a^x = M^k$, $a^y = M$

Therefore, $a^x = a^{ky} \Rightarrow x = ky$

Thus, $\log_a(M^k) = k\log_a M$

Change of Base Formula

Statement: $\log_a(M) = \dfrac{\log_b M}{\log_b(a)}$, For $b > 0, b \neq 1$

Proof: Let $x = \log_a(M) \Rightarrow a^x = M$

Now take log with base b on both the sides,

$\log_b(a^x) = \log_b M \Rightarrow x\log_b a = \log_b M$

Thus, $x = \log_a(M) = \dfrac{\log_b(M)}{\log_b(a)}$

Log of 1

Statement: $\log_a(1) = 0$

Proof: Let $x = \log_a 1 \Rightarrow a^x = 1$

Since, $a^0 = 1$, for any $a > 0, a \neq 1$

$a^0 = 1 = a^x \Rightarrow x = 0$

Thus, $x = \log_a(1) = 0$

Log of *a* number to the same base

Statement: $\log_a a = 1$

Proof: Let $x = \log_a a \Rightarrow a^x = a$

Since, $a^1 = a$, for any $a > 0, a \neq 1$

$a^1 = a^x \Rightarrow x = 1$

Thus, $x = \log_a a = 1$

Let us summarize the properties of logarithms we proved above in the table below:

For any base a, $a > 0$, $a \neq 1$ and M, N > 0

Property Name	Logarithmic Notation	What it does
Product Rule	$\log_a(M \times N) = \log_a M + \log_a N$	Turns multiplication into addition
Quotient Rule	$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$	Turns division into subtraction
Power Rule	$\log_a(M^k) = k \times \log_a M$	The exponent comes down
log of number to the same base	$\log_a(a) = 1$	Log of any number to the same base is 1
log of 1	$\log_a(1) = 0$	Logarithm of 1 to any base is zero
Base changing property	$\log_a n = \frac{\log_b n}{\log_b a}$	Base a is changed to any other base b ($b > 0$ and $b \neq 1$)

Remember: For positive values of m, n, x, y and $a > 0$, $a \neq 1$,

- $\log_a(m+n) \neq \log_a m + \log_a n$
- $\log_a(m-n) \neq \log_a m - \log_a n$
- If $x \neq y$ then $\log_a x \neq \log_a y$

Example 3: Write it as a single logarithm

(a) $\log_7 3 + \log_7 5$ (b) $\log_2 9 - \log_2 3$

(c) $\log_4 3 + \log_4 6 - 3\log_4 2$ (d) $1 + \log_3 5$

Solution:

(a)
$$\log_7 3 + \log_7 5 = \log_7(3 \times 5)$$
$$= \log_7 15$$

(b)
$$\log_2 9 - \log_2 3 = \log_2\left(\frac{9}{3}\right)$$
$$= \log_2 3$$

(c) $$\log_4 3+\log_4 6-3\log_4 2 = \log_4(3\times 6)-\log_4 2^3$$
$$= \log_4(18)-\log_4 8$$
$$= \log_4\left(\frac{18}{8}\right)$$
$$= \log_4\left(\frac{9}{4}\right)$$

(d) $$1+\log_3 5 = \log_3(3)+\log_3 5$$
$$= \log_3(3\times 5)$$
$$= \log_3 15$$

Example 4: Find the value of:

(a) $\log_7 343$ (b) $\log_3 27\sqrt{3}$

Solution:

(a) $$\log_7 343 = \log_7\left(7^3\right)$$
$$= 3\log_7(7) = 3$$

(b) $$\log_3 27\sqrt{3} = \log_3\left(3^3\times 3^{\frac{1}{2}}\right)$$
$$= \log_3 3^{\left(3+\frac{1}{2}\right)}$$
$$= \log_3 3^{\left(\frac{7}{2}\right)} = \frac{7}{2}$$

Example 5: Simplify: $\log_3 81-\log_3 9$

Solution: $$\log_3 81-\log_3 9 = \log_3 3^4-\log_3 3^2$$
$$= 4\log_3 3-2\log_3 3$$
$$= 4-2 = 2$$

2.4 Logarithm to base 10

Logarithms in base 10 are called **common logarithm** as they are used in many common scales, such as the **Richter scale** for measuring the magnitude of earthquakes, the **pH scale** for measuring acidity or alkalinity and the **decibel scale** for measuring sound.

$\log_{10} x$ is often written as just log x, and we *assume* the logarithm has base 10.

Rules of common logarithms:

The rules of common logarithm given below are similar to the one derived earlier:

- $\log(xy) = \log x + \log y$
- $\log\left(\frac{x}{y}\right) = \log x - \log y$
- $\log\left(x^n\right) = n \times \log x$
- $\log 1 = 0$
- $\log 10 = 1$

Example 6: Express the following as a single logarithm

(a) $\log 2 + \log 7$ (b) $\log 6 - \log 3$

Solution:

(a) $\log 2 + \log 7 = \log (2 \times 7)$

$= \log 14$

(b) $\log 6 - \log 3 = \log\frac{6}{3}$

$= \log 2$

Example 7: If $\log_3 7 = a$ and $\log_3 4 = b$. Write the following in terms of a and b.

(a) $\log_3\left(\frac{4}{7}\right)$ (b) $\log_3 28$ (c) $\log_3\left(\frac{7}{3}\right)$

Solution:

(a) $\log_3\left(\frac{4}{7}\right) = \log_3 4 - \log_3 7 = b - a.$

(b) $\log_3 28 = \log_3(4 \times 7) = \log_3 4 + \log_3 7 = b + a.$

(c) $\log_3\left(\frac{7}{3}\right) = \log_3 7 - \log_3 3 = a - 1.$

Example 8: Express the following as a single logarithm:

(a) $3 - \log_2 5$ (b) $1 + \log 2$

Solution:

(a) $3 - \log_2 5 = 3 \times 1 - \log_2 5$

Substituting $\log_2 2 = 1$ we get

$3 - \log_2 5 = 3 \times \log_2 2 - \log_2 5$

$= \log_2 2^3 - \log_2 5$

$$= \log_2 8 - \log_2 5$$

$$= \log_2 \frac{8}{5}.$$

(b) $1 + \log 2 = \log 10 + \log 2$ (as $\log_{10} 10 = 1$)

$$= \log (10 \times 2)$$

$$= \log 20$$

Note:

- Common logarithms have an interesting property of scaling positive numbers that are very small or very large. For example, if a certain quantity can take values from 0.0000000001 to 10,000,000,000 then the common logarithms of these numbers would lie in range –10 to 10.
- Natural logarithms are logarithm to the base e ($2 < e < 3$), where e like π is an irrational number. The natural logarithm are denoted by ln x i.e., $\log_e x = \ln x$.

EXERCISE 2.2

1. Express the following as a single logarithm:

   (a) $\log 2 + 2 \log 7$ (b) $\log_3 8 + \log_3 5 - \log_3 4$

   (c) $\log 5 + 2 \log 3 - \log 15$ (d) $2 + 2\log_5 3$

   (e) $3 - \frac{1}{2}\log_3 9$ (f) $1 + 2\log_4 3 - 3\log_4 4$

2. Find the exact value of

   (a) $\log_{11} 121$ (b) $\log_7 1$ (c) $\log_5 625$

   (d) $\log_8 8$ (e) $\log 1000$

3. If $\log_2 3 = p$ and $\log_2 5 = q$. Write the following in terms of p and q

   (a) $\log_2 15$ (b) $\log_2 45$

   (c) $\log_2\left(\frac{5}{3}\right)$ (d) $\log_2 10$

4. Which of the following are true?

   (a) If $2^{x+1} = 3^{x+2}$ then $x + 1 = x + 2$ (b) $\log (x + 1) = \log x$

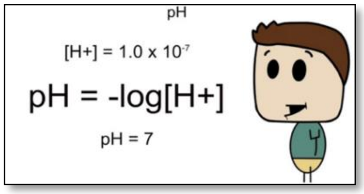
   (c) $\log_b b^3 = 3$ (d) Logarithm to base 1 is not defined.

5. If $\log_{2026} x - \log_{2026} y = a$, $\log_{2026} y - \log_{2026} z = b$ and $\log_{2026} z - \log_{2026} x = c$, then find the value of $\left(\frac{x}{y}\right)^{b-c} \times \left(\frac{y}{z}\right)^{c-a} \times \left(\frac{z}{x}\right)^{a-b}$.

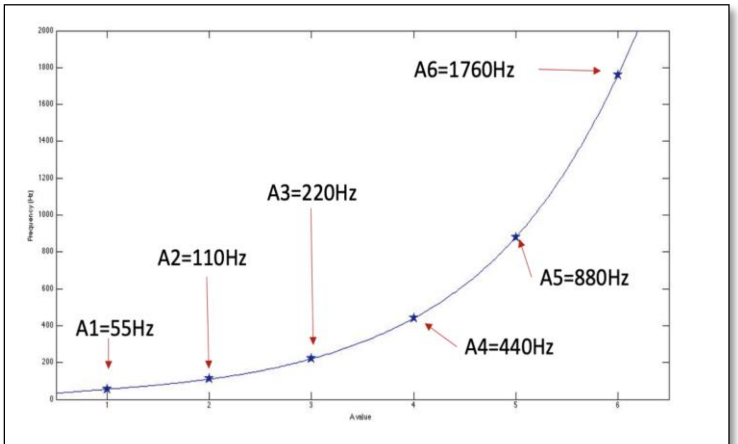
2.5 Logarithms Across Subjects

Logarithms are the "mathematical tools" used in Science, Music, and even Social science to manage scales that grow too fast such as population or earthquakes.

1. **Chemistry (The pH Scale):** Scientists measure how acidic a liquid is using the concentration of Hydrogen ions. Because these numbers are very small (like 0.00001), they use a negative log scale: $pH = -\log_{10}[H^+]$

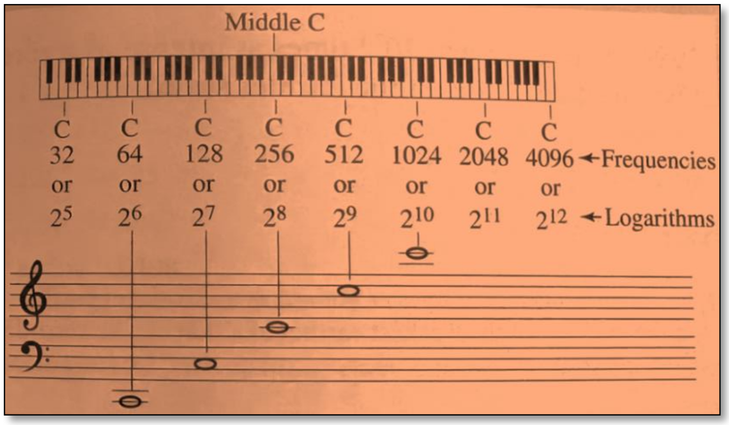


2. **Music (The Octave):** When a musician goes up one octave, the frequency of the sound doubles (2^1, 2^2, 2^3) but we perceive it as equal steps (1, 2, 3). It is interesting to learn that our ears hear sound logarithmically.



octave frequencies graph

The above graph shows all the A-note octaves from A1 to A6.



Note that frequency of each C note is written as powers of 2.

3. **Social Science:** Population growth often follows exponential patterns: $P = P_0(1+r)^t$. Logarithms help calculate **time required for population to double**.

4. **Geography (Earthquakes):** The Richter scale used to measure earthquakes is logarithmic. A magnitude 7 earthquake is not "two units" stronger than a magnitude 5 earthquake; it is 10^2 (100 times) more intense in amplitude!

Example 9: An earthquake measures 2 on the Richter scale, while another earthquake measures 5 on the same scale. How many times stronger is the second earthquake than the first?

Solution: Richter scale difference:

$$5 - 2 = 3$$

On the Richter scale, each 1-unit increase means **10 times stronger**.

So a 3-unit increase means:

$$10^3 = 1000$$

The magnitude 5 earthquake is **1000 times** stronger than the magnitude 2 earthquake.

EXERCISE 2.3

1. Express the following in logarithmic form:

   (a) $5^4 = 625$ (b) $10^{-2} = 0.01$

   (c) $7^0 = 1$ (d) $8^1 = 8$

2. Using the properties of logs, simplify: $\log_2 16 + \log_2 4$

3. Evaluate:

   (a) $\log_2 256$ (b) $\log_4 16$

   (c) $\log_5 125$ (d) $\log_{10} 0.001$

4. If $\log_2 7 = p$ and $\log_2 3 = q$. Write in terms of p and q.

   (a) $\log_2 21$ (b) $\log_2 49$

   (c) $\log_2\left(\frac{7}{3}\right)$ (d) $\log_2 63$

5. Real-world Application:

   (a) If a star is 100 times brighter than another, and the difference in their magnitudes is given by $2.5 \times \log_{10}$(brightness ratio), find the magnitude difference.

(b) A solution has pH 3 and another has pH 6. How many times more acidic is the first solution? ($pH = -\log_{10}[H^+]$)

(c) A magnitude 9 earthquake occurs on the Richter Scale. How many times stronger is it than a magnitude 4 earthquake?

6. **True or False:** (Explain your reasoning)

(a) $\frac{1}{3}\log_b x = \sqrt[3]{x};\ x > 0$

(b) $\log_8 e = \frac{1}{\ln 8}$

(c) Logarithm of a negative number is defined.

(d) $\log_b(M+N) = \log_b M + \log_b N$

(e) The base of the logarithm can be any real number.

7. Which is the greatest integer that is less than the number $\log_4 9 + \log_9 28$? (Do not use calculator)

8. Evaluate the value of $(x + 5y)$, where, $x = \log_{1.4\bar{3}}\left(\frac{43}{40}\right)$ and $y = \left(\frac{1}{2}\right)^{\log_2 5}$.

2.6 Solving Logarithmic Equations: The Search for '*x*'

In our journey through logarithms, we have seen how they help us rethink the relationship between numbers and exponents. However, the true power of a logarithm is revealed when it becomes an active tool for solving equations where the unknown value, x, is trapped within a logarithm or a base.

Solving a logarithmic equation is much like being a mathematical detective. You must use the Product, Quotient, and Power rules to combine multiple logarithms into a single expression, and then convert it into its exponential form to find the value of the variable.

The Golden Rule of Logarithmic Equations: You must always verify your solutions! Because the domain of a logarithmic function is strictly positive, the value of $\log_b a$ is defined if a is positive ($a > 0$), and base (b) must be positive and not equal to 1 ($b > 0$, $b \neq 1$). Sometimes, standard algebraic steps will produce an **extraneous root**—a false solution that mathematically breaks these rules. If substituting your answer back into the original equation results in the logarithm of a negative number or zero, that solution must be rejected.

We will understand this through the following solved examples.

Example 10: Solve for $x : \log_2(3x-1) = 3$.

Solution: $\log_2(3x-1) = 3$

Converting logarithmic equation into its exponential form,

$$3x - 1 = 2^3$$

$$\Rightarrow \quad 3x = 9$$

$$\Rightarrow \quad x = 3$$

Check: Substitute $x = 3$ in $\log_2(3x-1)$ gives $\log_2 8$. Since $8 > 0$, the logarithm is defined.

$\therefore$ $x = 3$ is the required solution.

Example 11: Solve for $x : \log_5 x + \log_5(x-4) = 1$

Solution: Using the product rule, we get

$$\log_5[x(x-4)] = 1$$

$$\Rightarrow \quad x(x-4) = 5^1$$

$$\Rightarrow \quad x^2 - 4x - 5 = 0$$

$$\Rightarrow \quad (x-5)(x+1) = 0$$

$$\Rightarrow \quad x = 5, -1$$

Note that on substituting $x = -1$, in the equation gives $\log_5(-1)$ and $\log_5(-5)$. Since log of a negative number is not defined so $x = -1$, is an extraneous root which is rejected.

The only valid solution is $x = 5$.

Example 12: Solve for $x : \log_3(x^2 - 8x) = 2$

Solution: $\log_3(x^2 - 8x) = 2$

Converting logarithmic equation into its exponential form,

$$x^2 - 8x = 3^2$$

$$\Rightarrow \quad x^2 - 8x - 9 = 0$$

$$\Rightarrow \quad (x-9)(x+1) = 0$$

$$\Rightarrow \quad x = 9, -1$$

Note that for both $x = 9, -1$, the expression $(x^2 - 8x)$ evaluates to 9, which is strictly greater than zero. Therefore, both $x = 9, -1$ are valid solutions.

Example 13: Solve for $x : \log_2(3x-4) = \log_2 5$

Solution:
$$\log_2(3x-4) = \log_2 5$$

$\Rightarrow$ $$3x - 4 = 5 \qquad (\text{as } \log_a x = \log_a y \Rightarrow x = y)$$

$\Rightarrow$ $$3x = 9$$

$\Rightarrow$ $$x = 3$$

Example 14: Solve for $x : (\log_3 x)^2 - 5(\log_3 x) + 6 = 0$

Solution: $(\log_3 x)^2 - 5(\log_3 x) + 6 = 0$

Let $y = \log_3 x,$ the given equation reduces to

$$y^2 - 5y + 6 = 0$$

$\Rightarrow$ $$(y-3)(y-2) = 0$$

$\Rightarrow$ $$y = 3 \text{ or } y = 2$$

$\therefore$ $$\log_3 x = 3 \text{ or } \log_3 x = 2$$

$\Rightarrow$ $$x = 3^3 \text{ or } x = 3^2$$

i.e. $$x = 27, 9$$

For both $x = 27$, and $x = 9$, the value of $\log_3 x$ is defined as $x > 0$, so required solutions are $x = 9, 27$.

Example 15: Solve for $x : \log_b(\log_b Ax) = 1;\ A > 0$

Solution:
$$\log_b(\log_b Ax) = 1$$

$\Rightarrow$ $$\log_b Ax = b$$

$\Rightarrow$ $$Ax = b^b$$

$\Rightarrow$ $$x = \frac{1}{A}(b^b)$$

EXERCISE 2.4

1. Solve for x.

   (a) $\log_3(2x-5) = 2$ (b) $\log_7(3x) + \log_7 2 = \log_7 24$

   (c) $\log_5(x+3) - \log_5(x-1) = 1$ (d) $\log_2(x^2-7) = 3$

2. Solve for x.

   (a) $\log_2(x-3) + \log_2(x+1) = 5$ (b) $2\log_4 x = \log_4(5x-4)$

   (c) $\log_5(x+2) + \log_5(x-2) = 1$ (d) $\log_{10}(x-2) + \log_{10}(x+1) = 1$

3. Solve for x.

(a) $\log_x(3x+10) = 2$, where $x > 0$ and $x \neq 1$

(b) $(\log_3 x)^2 - 4\log_3 x + 3 = 0$

(c) $(\log_2 x)^2 + \log_2 x^3 = 10$

(d) $x^{\log_{10} x} = 1000x^2$

4. Solve for x.

(a) $\log_3(x^2-1) = \log_3(2x-1)$

(b) $\log_x 5 - \log_x 2 = \log_x \sqrt{x}$

(c) $\log_2 x + \dfrac{1}{\log_x 2} = 4$

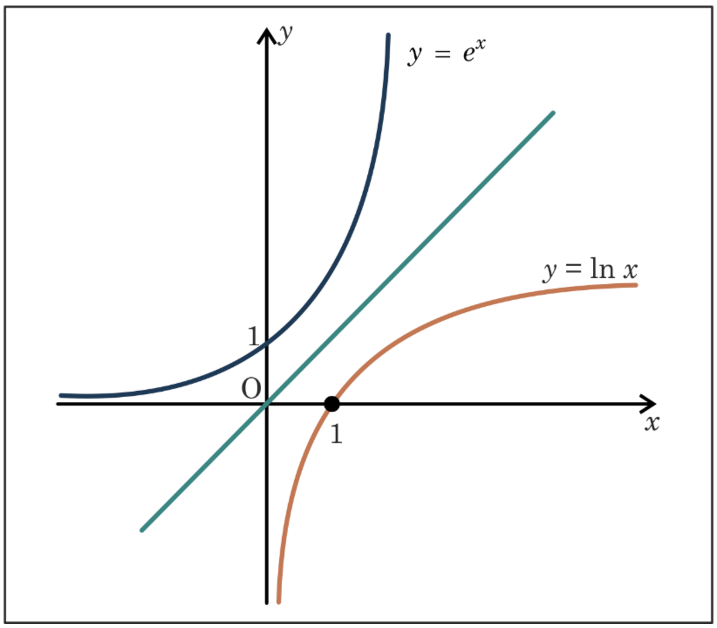
(d) $\log_3(3+x) + \log_3(8-x) - \log_3(9x-8) = 2 - \log_3 9$

(e) $\log_{10}\left[\log_2\left(\log_3 9\right)\right] = 5x$

5. If $x = \log\dfrac{1}{2} + \log\dfrac{2}{3} + \log\dfrac{3}{4} + \ldots + \log\dfrac{99}{100}$, where all logs are to the base 10, then evaluate $(x + 1)(x + 2)(x + 3) \ldots (x + 99)$.

Enrichment – Graph of logarithmic and Exponential functions

The graph of logarithmic function and exponential function are inverse of each other. Note that the graph of these functions are mirror images along the line $y = x$.



Summary

- **Logarithm** is the inverse operation of exponentiation.
- $\log_b a = x \Leftrightarrow b^x = a$, where $a > 0$ and $b > 0$, $b \neq 1$
- Logarithms are defined only for positive numbers.
- If the base of a logarithm is not given then we assume it to be 10 and such logarithms are called common logarithms.
- **Important Properties:**

  For $b > 0$, $b \neq 1$ and M > 0, N > 0 we have:

  - Product Rule: $\log_b(M \times N) = \log_b M + \log_b N$
  - Quotient Rule: $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$
  - Power Rule: $\log_b(M^k) = k \times \log_b M$
  - Log of a number to the same base: $\log_b(b) = 1$
  - Log of 1: $\log_b(1) = 0$
  - The base changing property:

    For any logarithmic bases b and a and any positive number n,

$$\log_b n = \frac{\log_a n}{\log_a b}$$

- **Applications:** Logarithms are widely used in:
  - Earthquake measurement
  - Acidity (pH scale)
  - Sound intensity
  - Population growth
  - Sports science

Chapter 3

Relations and Functions

3.1 Introduction

In everyday life, we often observe associations between two quantities. The marks obtained by a student in a MCQ test depends on the number of correct answers; the area of a circle depends on its radius; a person's salary may depend on the years of experience; the distance travelled at uniform speed may depend on the time taken etc.

All such dependencies or associations can be described mathematically using the concepts of relations and functions.

In a school each student is allotted one house. This rule of connection — where every student is paired with precisely one house is a function. If instead a student is associated with one or more sports teams, we simply have a relation. The concept of function is fundamental to all mathematics.

In this chapter, we will define ordered pairs, cartesian products, relations, some functions and their graphs, domain, range, vertical line test and some simple transformations.

3.2 Ordered Pairs

An ordered pair is a pair of objects written in a specific order. It is denoted by (a, b), where a is called the first element and b is the second element. Since the order matters, hence in general (a, b) is different from (b, a).

Definition of an Ordered Pair

An ordered pair (a, b) consists of two elements separated by comma and written in parenthesis i.e., ( ). In (a, b) the first element is a and the second is b. Two ordered pairs are equal if and only if both the corresponding components are equal, i.e.,

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d$$

Note that $(a, b) \neq (b, a)$ unless $a = b$. This distinguishes an ordered pair from a set $\{a, b\}$, where order is irrelevant. You may recall that an ordered pair (x, y) is used to represent the position of the point in the cartesian plane, where x is known as the abscissa and y is known as the ordinate.

Example 1: If $(2x + 1, y - 3) = (9, 4)$, find x and y.

Solution: Since the ordered pairs are equal, equating their abscissas and ordinates, we get

$2x + 1 = 9$	$y - 3 = 4$
i.e, $x = 4$	i.e, $y = 7$

3.3 Cartesian Product of Sets

Given two non-empty sets A and B, their cartesian product A × B (read as 'A cross B') is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

Definition of Cartesian Product

For any two sets A and B we have, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

If $A = \phi$ or $B = \phi$, then $A \times B = \phi$.

Properties of Cartesian product:

- $n(A \times B) = n(A) \times n(B)$.
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$ and $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
- If $A \subseteq B$ then $A \times C \subseteq B \times C$ for any set C.

Note: The cartesian product of two sets is not commutative i.e., $A \times B \neq B \times A$.

Example 2: Let A = {1, 2, 3} and B = {4, 5}. Find A × B and B × A. Are they equal?

Solution: $A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$

$B \times A = \{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}$

Since $(1, 4) \in A \times B$ but $(1, 4) \notin B \times A$, So, $A \times B \neq B \times A$.

Example 3: If $n(A) = 4$ and $n(B) = 3$, find how many subsets will A × B have?

Solution: We know that $n(A \times B) = n(A) \times n(B)$

$= 4 \times 3 = 12$

Number of subset of $A \times B = 2^{12} = 4096$.

Example 4: If A = {a, b} and B = {1, 2, 3}, write A × B and verify

$$n(A \times B) = n(A) \times n(B).$$

Solution:

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

$$n(A \times B) = 6$$

$$n(A \times B) = 2 \times 3$$

$$n(A \times B) = n(A) \times n(B).$$

Hence verified.

Example 5: If A = {1, 3, 5} and B = {4, 9}. Write the ordered pairs of A × B whose first element is less than the second element. What about the ordered pairs of B × A satisfying the same order relation? Are they equal?

Solution: The ordered pairs of A × B having first element less than the second element are (1, 4), (1, 9), (3, 4), (3, 9) and (5, 9). The ordered pair of B × A satisfying the same order relation is (4, 5) only. The elements of A × B and B × A satisfying the same rule are different.

Example 6: Let A = {1, 3}, B = {2, 3, 4} and C = {4, 5}. Write the set A × (B ∩ C) and verify A × (B ∩ C) = (A × B) ∩ (A × C).

Solution:

$$B \cap C = \{2, 3, 4\} \cap \{4, 5\}$$

$$B \cap C = \{4\}$$

$$A \times (B \cap C) = \{1, 3\} \times \{4\} = \{(1, 4), (3, 4)\}$$

$$(A \times B) \cap (A \times C) = \{(1, 2), (1, 3), (1, 4), (3, 2), (3, 3), (3, 4)\} \cap \{(1, 4), (1, 5), (3, 4), (3, 5)\}$$

$$= \{(1, 4), (3, 4)\}$$

So, $A \times (B \cap C) = (A \times B) \cap (A \times C)$

EXERCISE 3.1

1. If $(x - 5, y + 1) = (4, 6)$, find x and y.
2. Let A = {1, 2} and B = {2, 3, 5}. List all elements of A × B and B × A.
3. If $n(A \times B) = 20$ and $n(A) = 4$, find $n(B)$.
4. If A = {1, 2, 3} and B = $\{x, y\}$, find A × B, B × A, A × A and B × B.
5. If A = {1, 2, 3} and B = {2, 3, 7}, find (A × B) ∩ (B × A).
6. Verify, A × (B ∪ C) = (A × B) ∪ (A × C) for A = {1, 2}, B = {2, 3}, C = {4, 5}.

3.4 Relations

A relation constitutes only those elements of cartesian product which satisfies a rule that governs the relation between the elements of the two sets. Any subset of the cartesian product A × B is a relation from A to B. Let us give a formal definition of the relation.

Definition of a Relation

A relation R from set A to set B is a subset of A × B, i.e., R ⊆ A × B. If $(a, b) \in R$ we write a R b and read as 'a is related to b'. The set of all first elements of the ordered pairs in the relation constitutes the domain of R and the set of all second elements constitutes the range of R.

- **Domain of R:** The set of all first elements of the ordered pairs is the domain, i.e., Domain = $\{a \in A : (a, b) \in R$ for some $b \in B\}$. The domain of R is also called the set of pre-images
- **Range of R:** The set of all second elements of the ordered pairs is the range, i.e., Range = $\{b \in B : (a, b) \in R$ for some $a \in A\}$. The range of R is also called the set of images.
- **Codomain of R:** The entire set B is the codomain of the relation.

  Note:

  - range $\subseteq$ codomain.
  - A relation R on a set A means a relation from set A to A.

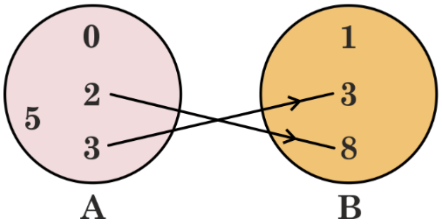
Let us represent the relation, $R : A \rightarrow B$ by an arrow diagram as shown below:

Note that, R = {(2, 8), (3, 3)}

therefore, Domain = {0, 2, 3, 5}

Range = {3, 8}

Codomain = B = {1, 3, 8}



Example 7: Let A = {1, 2, 3, 4}, B = {1, 4, 9, 20, 25}. A relation R from A to B defined by $R = \{(a, b) : a^2 = b; a \in A, b \in B\}$. Write the relation R in roster form, find its domain, range, and codomain.

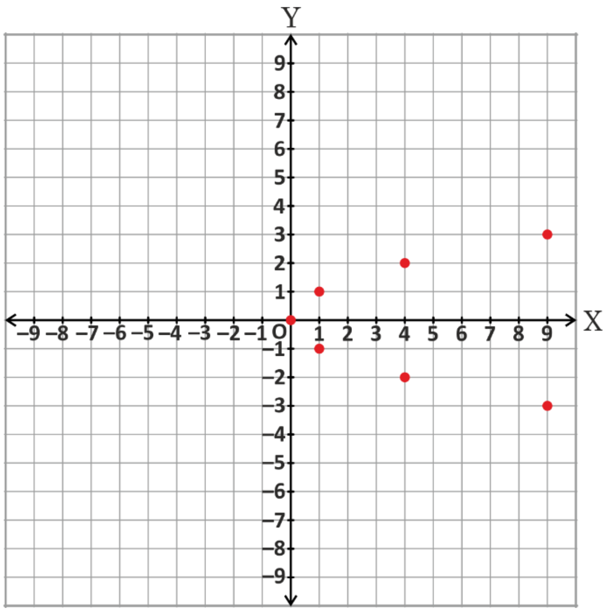
Solution: Since, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, therefore R = {(1, 1), (2, 4), (3, 9)}

Domain = {1, 2, 3}, Range = {1, 4, 9}, Codomain = {1, 4, 9, 20, 25}.

Example 8: What is the domain and range of the relation given below:

R = {(0, 0), (1, 1), (1, –1), (4, 2), (4, –2), (9, 3), (9, –3)}

Represent R on the graph. Also identify the rule that defines this relation.



Solution: Taking x coordinates we get,

Domain = {0, 1, 4, 9}

Taking y coordinates we get,

Range = {–3, –2, –1, 0, 1, 2, 3}

The relation R is given by the equation $y = \pm\sqrt{x}$ for $x = 0, 1, 4, 9$.

Example 9: Given A = {1, 3, 8}, B = {2, 3, 9, 11}

A relation, $R : A \rightarrow B$ defined by R = {(a, b) : $b \mid a$; $a \in$ A, $b \in$ B}

Write the relation R in roster form.

Also determine its domain, range and codomain.

(Note: $b \mid a$ means b is a factor of a or b divides a exactly)

Solution: In roster form, R = {(3, 3), (8, 2)}

Domain = {3, 8}

Range = {2, 3}

Codomain = B = {2, 3, 9, 11}

Example 10: How many relations can be defined from set A = {1,2,3} to B = {4, 5}?

Solution: Since, $n(A \times B) = 3 \times 2 = 6$.

Therefore, number of relations = $2^6 = 64$.

Number of subsets of a set consisting of n elements is 2^n.

Example 11: Gaurav studies in class XI in which there are 30 students. His sister studies in class IX having 20 students.

How many relations are possible from set A to set B if set A represents the set of 30 students of class XI and set B represents a set of 20 students of class IX?

Solution: We first find number of element in A × B

i.e. $n(A \times B) = 30 \times 20 = 600$

Now since relation is a subset of cartesian product, so number of relations is equal to the number of subsets of a set containing 600 elements i.e. 2^{600}

2^{600} is enormous, far beyond anything we encounter in everyday counting.

Example 12: Let us defined a relation S from Z to Z where

S = {(x, y) : difference between x and y is odd}

Write the domain and range of the relation.

Solution: The difference between an odd integer and an even integer is odd and the difference between an even integer and an odd integer is also odd i.e., (odd, even) and (even, odd) $\in$ S.

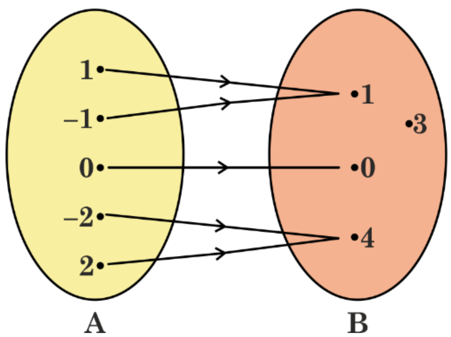
So, both domain and range of the relation S are the set of integers Z.

EXERCISE 3.2

1. Let A = {1, 2, 3}, B = {4, 5, 6, 7}. Define a relation R from A to B by R = {(a, b) : $a + b = 7$; $a \in$ A, $b \in$ B}. Write the relation R in roster form and hence find its domain and range.
2. Given A = {2, 3, 4, 5}, B = {3, 6, 7, 10}. Define R = {(a, b) : a divides b; $a \in$ A, $b \in$ B}. Write R in roster form hence find its domain and range.
3. Let R = {(a, b) : $a + 2b = 12$, a, $b \in$ N}. Write R in roster form and hence find its domain and range.
4. Write R = {(x, x^2) : x is a prime number less than 10} in roster form. Also find the range of R.
5. Let A = {p, q, r, s} and B = {1, 2}. How many relations can be defined from set A to set B? List any four of them.
6. Let A = {1, 2, 3, 4, 5}. Define a relation R on A by R = {(a, b) : $|a - b| = 2$}. Write R in roster form and hence find its domain and range.

3.5 Functions

A relation in which each element of the domain corresponds to exactly one element of the range is called a function.



The relation from set A to B, shown by the arrow diagram is also a function $f : A \to B$.

Note: Consider a function $f : A \to B$, where $f(a) = b$ then b is called the image of a under f and a is called the preimage of b under f.

Let us give formal definition of function.

Definition: A relation R from set A to set B is a function from A to B if every element of A has exactly one image in set B.

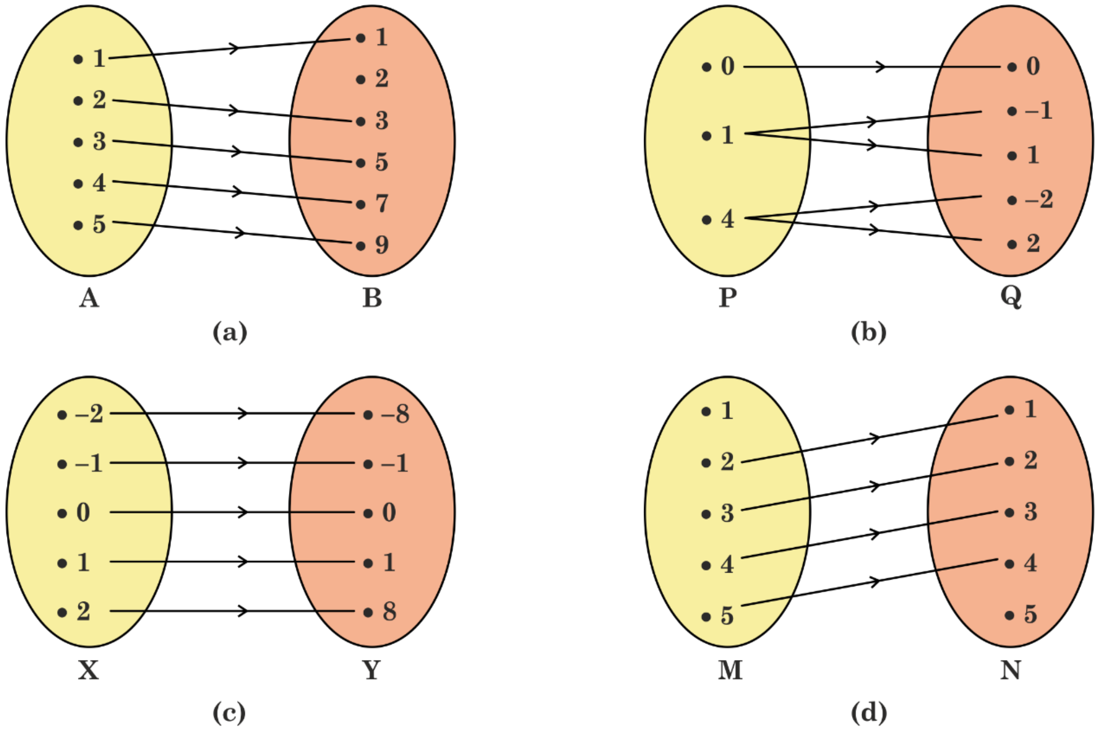
In other words a relation $R : A \to B$ is a function if the following two conditions are satisfied.

(i) every element of A has an image i.e. domain of R is A.

(ii) no element in A has more than one image i.e., the image is unique.

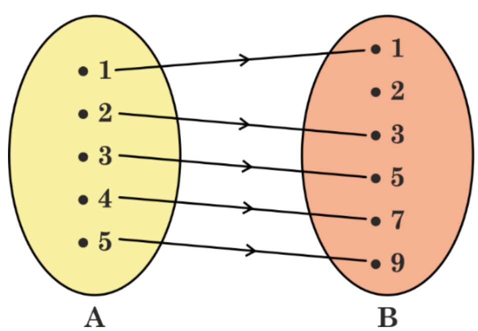
Note: A relation from set A to set B is a function if no two distinct ordered pairs have the same first element.

Example 13: Which of the following relations are functions? Justify your answer.

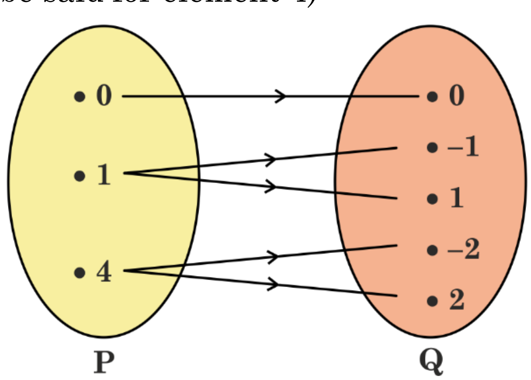


Solution:

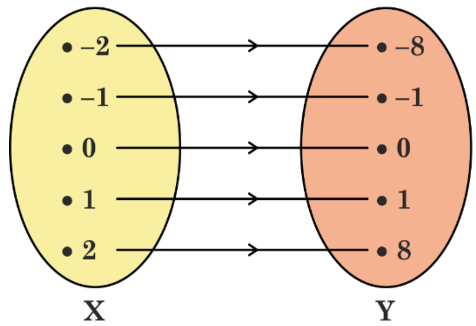
(a) The relation from A to B is a function as every element of set A has a unique image.



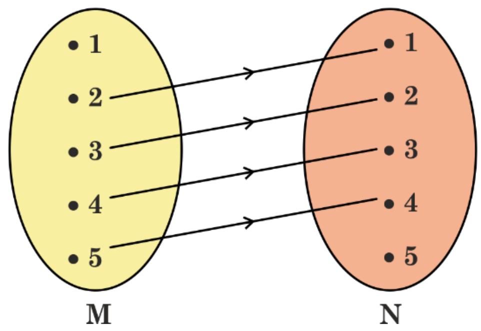
(b) The relation from P to Q is not a function as an element of P i.e., 1 has two images. (Same can be said for element 4)



(c) The relation from X to Y is a function as every element of set X has a unique image.



(d) The relation from M to N is not a function as an element of M i.e., 1 has no image.



Example 14: Determine which of the following rules describes a function. Give reason for each.

	Domain	Rule	Range
(a)	The set of students in a school	Teacher of each student.	The set of teachers in a school
(b)	The set of sportspersons participating in Olympics.	Each sportsperson country.	The set of countries.
(c)	The set of real numbers.	Square of a number.	The set of positive real numbers.
(d)	The set of real numbers.	Cube of a number.	The set of real numbers.
(e)	The set of integers.	Square root of a number.	The set of non-negative real numbers.

Solution: (a) No — A student is taught by different teachers. Since the association is not unique so it is **not** a function.

(b) Yes — Since each sportsperson represents his/her country, so the association is unique. Hence the rule describes a function.

(c) No — The square of 0 is 0, which is not there in set of positive real numbers. Since every element of domain does not have an image so the rule does **not** describe a function.

(d) Yes — The cube of every real number is again a unique real number, so the rule describes a function.

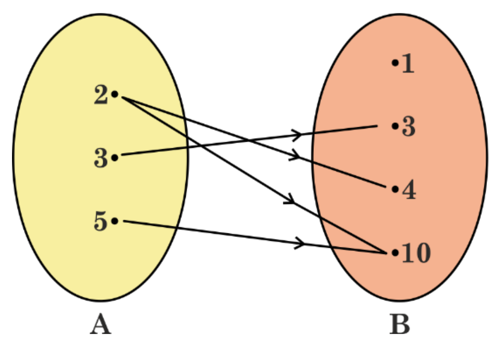
(e) No — The square root of a negative integer is not defined. Since negative integers does not have an image so the rule does **not** describe a function.

Example 15: Let A = {2, 3, 5}, B = {1, 3, 4, 10}

A relation $R : A \to B$ defined by $R = \{(a, b) : \text{iff } a \mid b\ ;\ a \in A,\ b \in B\}$. Write the relation in roster form and also draw its arrow diagram. Is R a function?

Also write its domain, co-domain and range.

Solution: Let us draw an arrow diagram of the given relation.



Since the element 2 of A has two images, so R is not a function.

We have, Domain = {2, 3, 5}

also, Range = {3, 4, 10}

and Co-domain = B = {1, 3, 4, 10}

Example 16: Which of the following relations from A = {1, 2, 3} to B = {a, b, c, d} are functions from A to B?

(i) $R_1 = \{(1, a), (2, b), (3, c)\}$ (ii) $R_2 = \{(1, a), (1, b), (2, c), (3, d)\}$

(iii) $R_3 = \{(1, a), (2, a), (3, a)\}$ (iv) $R_4 = \{(1, a), (2, c)\}$

Solution: (i) R_1 is a function — every element of A has exactly one image.

(ii) R_2 is not a function — element 1 has two images a and b.

(iii) R_3 is a function — every element of A has exactly one image.

(iv) R_4 is not a function — element 3 in A has no image.

Example 17: Let $f : \{1, 2, 3, 4\} \to N$ be defined by $f(x) = x^2 + 1$. Find the range of f.

Solution: $f(1) = 2,\ f(2) = 5,\ f(3) = 10,\ f(4) = 17$.

Range of $f = \{2, 5, 10, 17\}$.

Example 18: Let A = {2, 4, 9, 16, 25, 36}. Define a relation R on A i.e. from set A to A by R = $\{(x, y) : x = y^2\}$. Write its domain, co-domain and range.

Solution:

$$R = \{(4, 2), (16, 4)\}$$

$$\text{Domain} : x \in \{4, 16\}$$

$$\text{Range} : y \in \{2, 4\}$$

$$\text{Co-domain} = A = \{2, 4, 9, 16, 25, 36\}$$

Example 19: Let A = {1, 2, 3}. Define the following relations on A (i.e., from A to A).

(a) $\{(x, y) : y = x \text{ where } x, y \in A\}$

(b) $\{(x, y) : x + y \leq 6 \text{ where } x, y \in A\}$

(c) $\{(x, y) : y = 4x^3 \text{ where } x, y \in A\}$

Solution:

(a) Since, A = {1, 2, 3}

Therefore, R = {(1, 1), (2, 2), (3, 3)}

This is an identity relation.

(b) R = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3) (3, 1), (3, 2), (3, 3)}

The relation so obtained is a universal relation as it is equal to A × A.

(c) R = { }

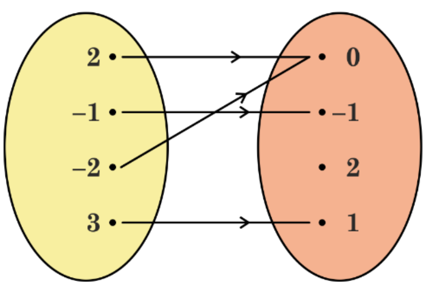
Since there is no ordered pair (a, b) which satisfies $b = 4a^3$.

So we get an empty relation.

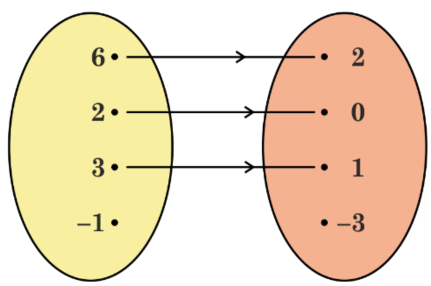
EXERCISE 3.3

1. Which of the following relations are functions? Justify your answer.

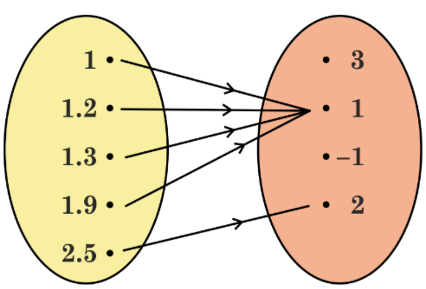
(a)



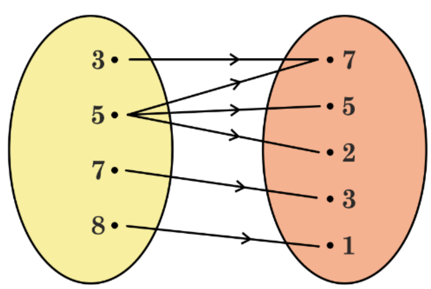
(b)



(c)



(d)



2. Which of the following relations from A = {3, 5, 7, 9} to B = {1, 2, 3, 4, 5} are functions from A to B?

(a) $R_1 = \{(3, 2), (5, 4), (7, 5), (9, 5)\}$

(b) $R_2 = \{(1, 3), (3, 5), (5, 7)\}$

(c) $R_3 = \{(2, 3), (2, 5), (2, 7), (3, 5), (3, 7), (5, 7)\}$

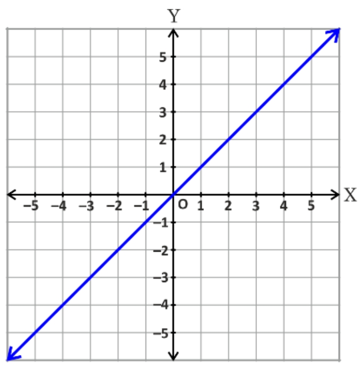
(d) $R_4 = \{(3, 3), (5, 5)\}$

3.6 Some Functions and their Graphs

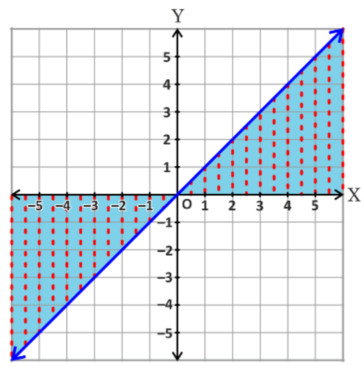
Let us draw graphs of some real valued functions. We also learn how to find the domain and range of a function by the projection of the graph on the x-axis and y-axis respectively. A projection of a curve on an axis is the set of points i.e. the foot of the perpendiculars drawn from every point on the curve onto the axis. For real functions projection is either R or its subset.

(a) Identity Function: A function $f : \mathrm{R} \to \mathrm{R}$, defined by $f(x) = x$.

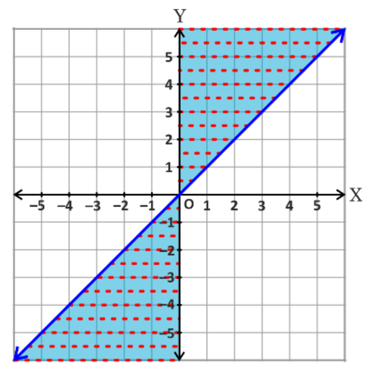
The projection of a curve along the x-axis is called the domain and the projection of the curve along the y-axis is called the range.



Graph of $f(x) = x$

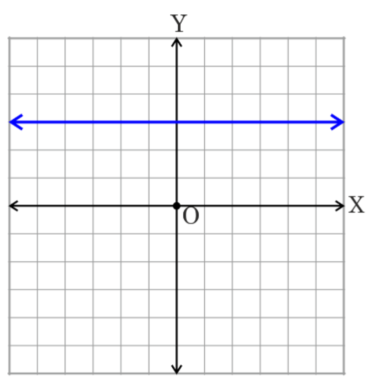


Domain : $x \in (-\infty, \infty)$ i.e., R

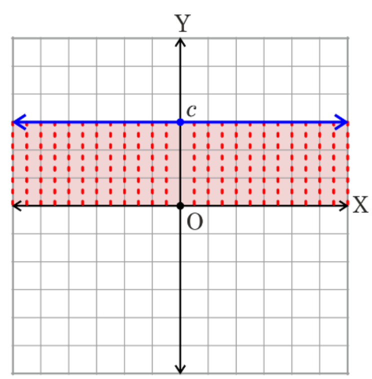


Range : $y \in (-\infty, \infty)$ i.e., R

(b) Constant Function: A function $f : \mathrm{R} \to \mathrm{R}$, defined by $f(x) = c$.



Graph of $f(x) = c$

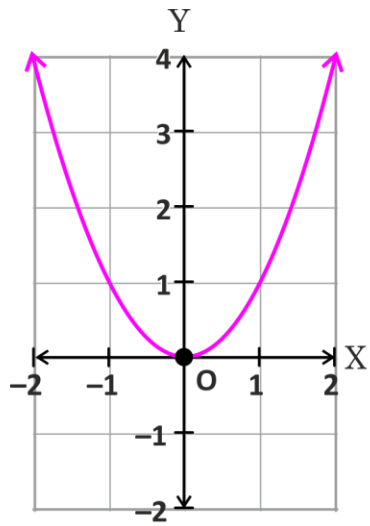


Domain : $x \in (-\infty, \infty)$ i.e., R

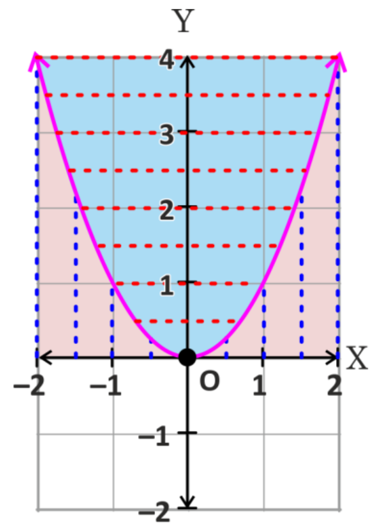
Range : $y \in \{c\}$

Note: The foot of the perpendicular of every point on the graph of $f(x) = c$ on the y-axis is $(0, c)$. So the range of the function is the set $\{c\}$.

(c) Quadratic Function: A function $f : \mathrm{R} \to \mathrm{R},$ defined by $f(x) = x^2$.



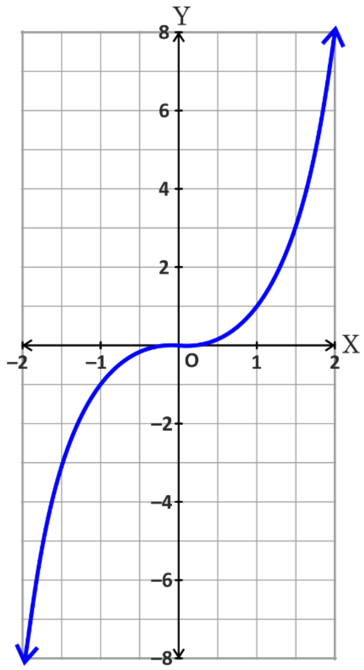
Graph of $f(x) = x^2$



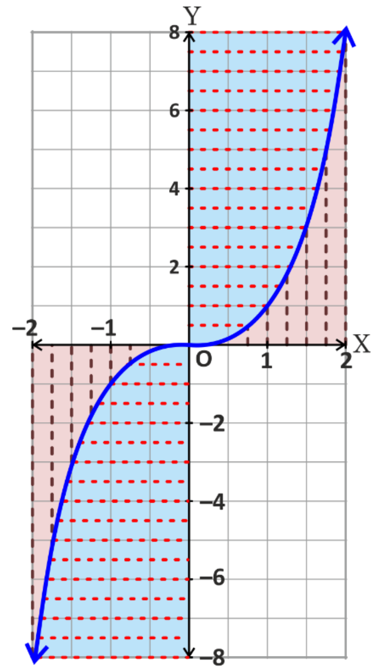
Domain : $x \in (-\infty, \infty)$ i.e., R

Range : $y \in [0, \infty)$

(d) Cubic Function: A function $f : \mathrm{R} \to \mathrm{R},$ defined by $f(x) = x^3$.



Graph of $f(x) = x^3$



Domain : $x \in (-\infty, \infty)$ i.e., R

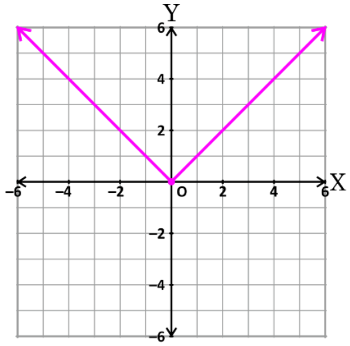
Range : $y \in (-\infty, \infty)$ i.e., R

Note:

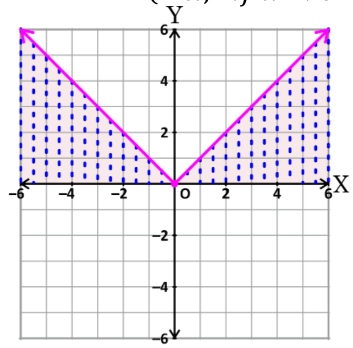
The coloured region is not part of the graph. It is drawn to show continuum of the projection. In fact infinitely many perpendicular lines can be drawn from the graph to the axes.

(e) Modulus Function: A function $f : R \to R$, defined by $f(x) = |x|$.

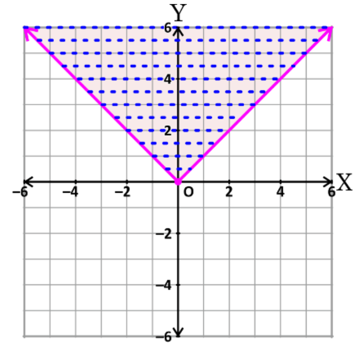
We have, $|x| = \begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{cases}$



Graph of $f(x) = |x|$



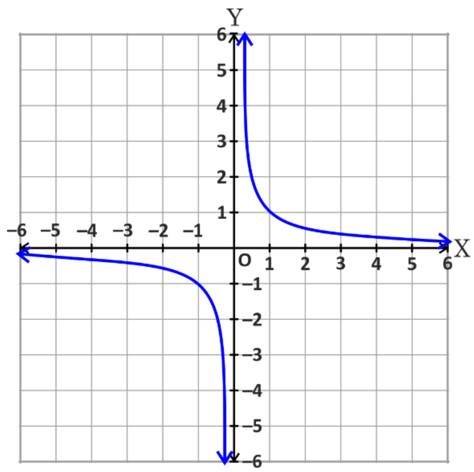
Domain : $x \in (-\infty, \infty)$ i.e., R



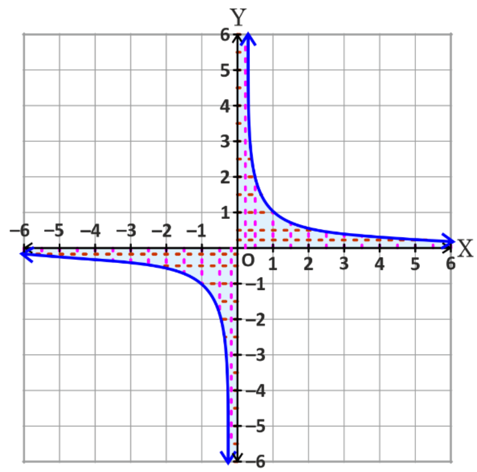
Range : $y \in [0, \infty)$

(f) Reciprocal Function: A function $f : R - \{0\} \to R$, defined by $f(x) = \frac{1}{x}$.

The graph of f is shown below:



Graph of $f(x) = \frac{1}{x}$

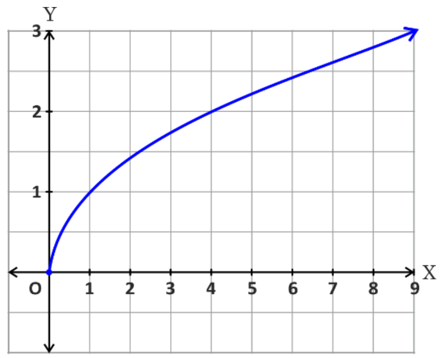


Domain : $x \in (-\infty, \infty) - \{0\}$

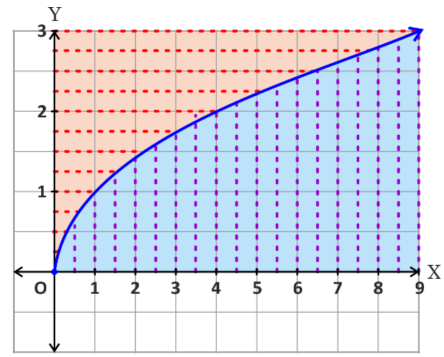
Range : $y \in (-\infty, \infty) - \{0\}$

(g) Square Root Function: The function $f : [0, \infty) \to R$, defined by $f(x) = \sqrt{x}$.

The graph of f is given below:



Graph of $f(x) = \sqrt{x}$



Domain : $x \in [0, \infty)$

Range : $y \in [0, \infty)$

(h) Greatest Integer Function: The function $f : R \to R$, defined by

$$f(x) = [x],\ x \in R$$

is the value of the greatest integer less than or equal to x.

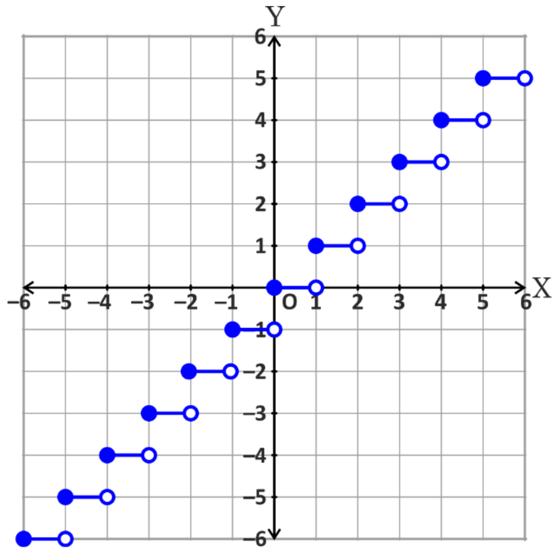
x	0	0.01	0.5	0.9	1	1.2	1.8	2	–1.2	–2
y	0	0	0	0	1	1	1	2	–2	–2

Note: From the definition of greatest integer function it follows:

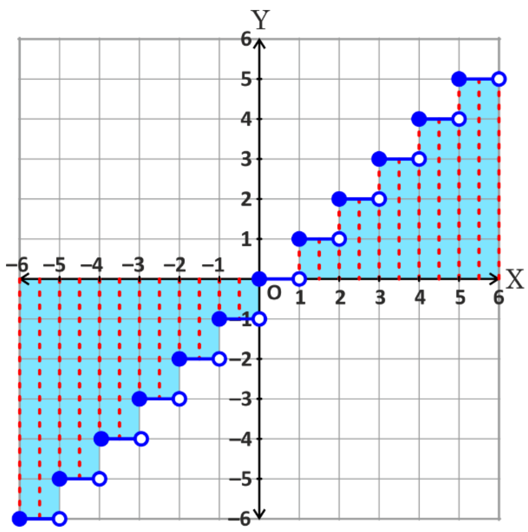
$[x] = 0$ for $0 \le x < 1$

$[x] = -1$ for $-1 \le x < 0$

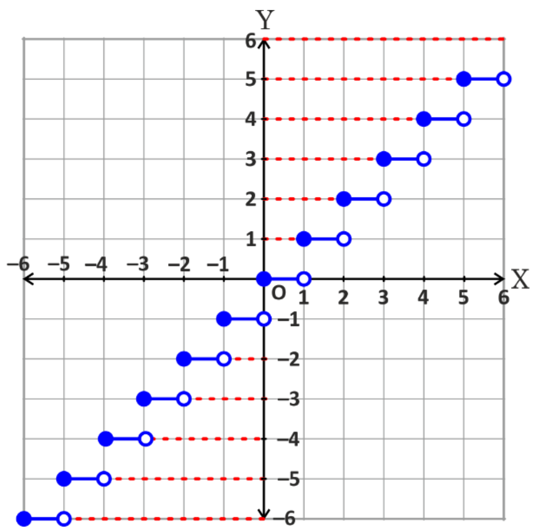
$[x] = 1$ for $1 \le x < 2$ and so on.



$f(x) = [x]$



Domain : $x \in (-\infty, \infty)$ i.e., R



Range : $y \in Z$

Note: Students are advised to take different values of x and find the corresponding values of y in tabular form and plot the point obtained to obtain the graph.

Example 20: Evaluate:

(i) $|-7| + |3|$ (ii) $[4.9] - [-2.3]$

(iii) $\left|\sqrt{|-25|} \div [-4.01]\right|$ (iv) $[0.0001] - [1.0001]$

Solution: (i) $7 + 3 = 10$ (ii) $4 - (-3) = 7$

(iii) $|5 \div (-5)| = 1$ (iv) $0 - 1 = -1$

Example 21: Draw the graphs of the functions f, g and h on the same coordinate axes. You may fill the tables given below to draw the graph:

$f(x) = x^2$,

x	$y = x^2$
–2	
–1	
0	
1	
2	

$g(x) = x^2 - 1$,

x	$y = x^2 - 1$
–2	
–1	
0	
1	
2	

$h(x) = x^2 + 1$

x	$y = x^2 + 1$
–2	
–1	
0	
1	
2	

Describe the relationship among the graph of f, g and h. Can you generalise the relationship among the graphs of $f(x)$, $f(x) - c$ and $f(x) + c$?

Solution: Let us fill the tables

$f(x) = x^2$,

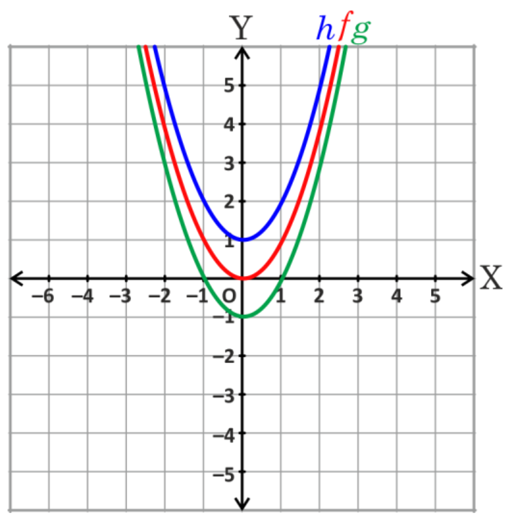
x	$y = x^2$
–2	4
–1	1
0	0
1	1
2	4

$g(x) = x^2 - 1$,

x	$y = x^2 - 1$
–2	3
–1	0
0	–1
1	0
2	3

$h(x) = x^2 + 1$

x	$y = x^2 + 1$
–2	5
–1	2
0	1
1	2
2	5



The graph of $g(x) = x^2 - 1$ is obtained by a vertical shift (downwards) of 1 unit below the x-axis and graph of $h(x) = x^2 + 1$ is obtained by a vertical shift (upwards) of 1 unit above the x-axis.

In general $y = f(x) - c$ where $c > 0$ represents a vertical translation (shift) of c units downwards and $y = f(x) + c$ represents a vertical translation (shift) of c units upwards. The shape of the curve remains identical.

EXERCISE 3.4

1. What is the domain and range of each of the relations given below? Which of these relations are functions:

(a) R = {(5, 1), (4, 1), (3, 1), (2, 0)}

(b) R = {(1, –1), (2, –2), (3, –3), (4, –4), (5, –5)}

(c) R = {(3, –1), (3, 0), (3, 1), (3, 2)}

2. Draw a rough sketch of each of the following relations. Also write their domain and range.

(a) $R = \{(x, y) : xy = 8 \text{ where } x, y \in Z\}$

(b) $R = \{(x, y) : x = |y| \text{ where } x \in Z \text{ and } 0 \leq x \leq 5\}$

(c) $R = \{(x, y) : y = -\sqrt{x} \text{ where } x \in (0, \infty)\}$

3. Draw the graph of the functions f, g and h on the same coordinate axes. You may fill the tables given below to draw the graphs.

$f(x) = |x|$,

x	$y = \|x\|$
–2	
–1	
0	
1	
2	

$g(x) = |x| - 1$,

x	$y = \|x\| - 1$
–2	
–1	
0	
1	
2	

$h(x) = |x| + 1$

x	$y = \|x\| + 1$
–2	
–1	
0	
1	
2	

Describe the relationship among the graphs of f, g and h.

4. Draw the graphs of the functions f, g and h on the same coordinate axes. You may fill the tables given below to draw the graph.

$f(x) = x^2$,

x	$y = x^2$
–2	
–1	
0	
1	
2	

$g(x) = (x - 1)^2$,

x	$y = (x - 1)^2$
–2	
–1	
0	
1	
2	

$h(x) = (x + 2)^2$

x	$y = (x + 2)^2$
–2	
–1	
0	
1	
2	

Describe the relationship among the graphs of f, g and h. Are their domain and range equal?

5. Determine the domain and range of the following functions:

(a) $y = \frac{1}{x^2}$

(b) $y = 2 - |x|$

(c) $y = (x - 1)^3$

(d) $y = \sqrt{-x}$

Summary

1. An ordered pair is a pair of objects generally numbers or variables written in specific order.

   For example: (3, –5), (x, y) etc.

2. The Cartesian product of two non-empty sets A and B is the set of all ordered pairs (a, b) where $a \in$ A and $b \in$ B.

   i.e., $A \times B = \{(a, b) : a \in A, b \in B\}$

3. A relation $R : A \rightarrow B$ is a subset of the cartesian product A × B. The set of x-coordinates of all ordered pairs constitute a domain and set of y-coordinates of all the ordered pairs constitute the range.

4. A relation $f : A \rightarrow B$ is a function if every element of set A has a unique image.

   The set A is called the domain and set of all images b where $f(a) = b$ is called the range.

5. The graph of $y = f(x) + c$ can be obtained from the graph of $y = f(x)$ by shifting it 'c' units above the x-axis if $c > 0$ and shifting it 'c' units below the x-axis if $c < 0$.

   The graph of $y = f(x + k)$ can be obtained from the graph of $y = f(x)$ by a horizontal shift of k units. If $k > 0$ the graph shifts to the left and if $k < 0$ the graph shifts to the right.

Chapter 4 Coordinate Geometry

4.1 Introduction: The Language of Graphs

Imagine trying to describe the exact location of a single star in the sky or a specific seat in a massive stadium. To do this accurately, we need a reference system. In mathematics, this system is Coordinate Geometry (also known as Analytical Geometry), where we use numbers to represent positions on a plane. Coordinate geometry is the study of geometry using a coordinate system. It bridges algebra and geometry.

4.2 The Cartesian System

The foundation of this chapter is the Cartesian Plane, named after the mathematician René Descartes.

The "Legend" of the Fly

The most famous story associated with René Descartes (1596–1650) is that the idea for coordinate geometry came to him while he was lying in bed watching a fly crawl on the ceiling. He realized that he could describe the fly's exact position at any moment by measuring its distance from two perpendicular walls.

Source of the image: https://learnodo-newtonic.com

While this story might be apocryphal, it perfectly illustrates the shift in thinking: **position can be defined by numbers**.

Now that we know René Descartes' big idea, let's look at the "map" he created. To locate a point, we need a frame of reference. For that, following terms are defined:

The Cartesian plane: Formed by the intersection of a horizontal line (x-axis) and a vertical line (y-axis) at the origin (0, 0). It divides the plane into four quadrants.

Coordinates: An ordered pair (x, y) represents any point P, where x is the abscissa (horizontal distance) and y is the ordinate (vertical distance) *. (*Refer to the graph on the next page)

DO YOU KNOW

Cartesian: This term is derived directly from the Latin version of Descartes' name, *Renatus Cartesius*.

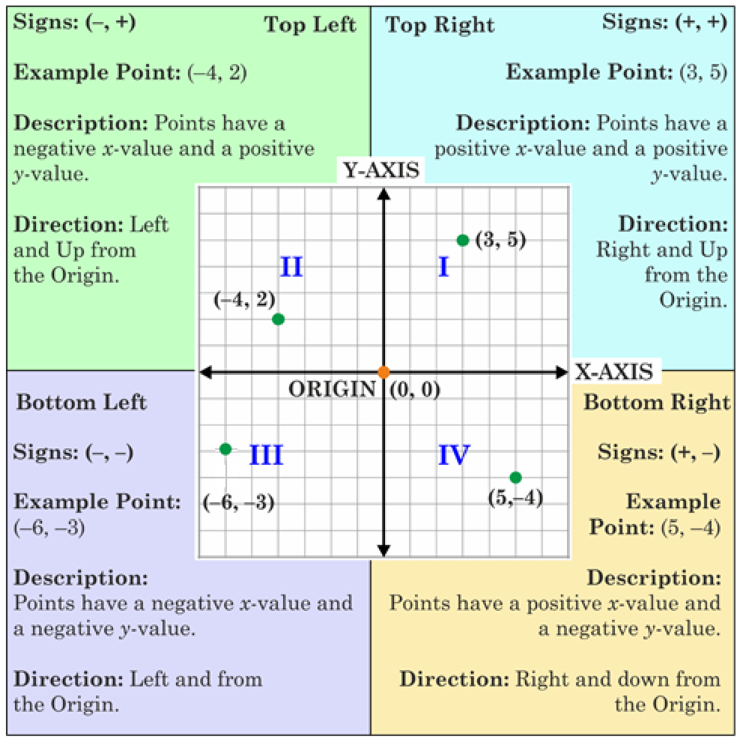
Coordinates: From the idea of "ordering" or "arranging" points in a mutual relationship to two axes.

4.3 Let Us Explore Four Quadrants

The Four Quadrants

These two axes (x-axis and y-axis) divide the entire plane into four regions called **Quadrants**. We number them anti-clockwise starting from the top right:

- **Quadrant I:** Both x and y are positive (+, +).
- **Quadrant II:** x is negative, y is positive (–, +).
- **Quadrant III:** Both x and y are negative (–, –).
- **Quadrant IV:** x is positive, y is negative (+, –).



4.4 Moving Points: The Magic of Reflections

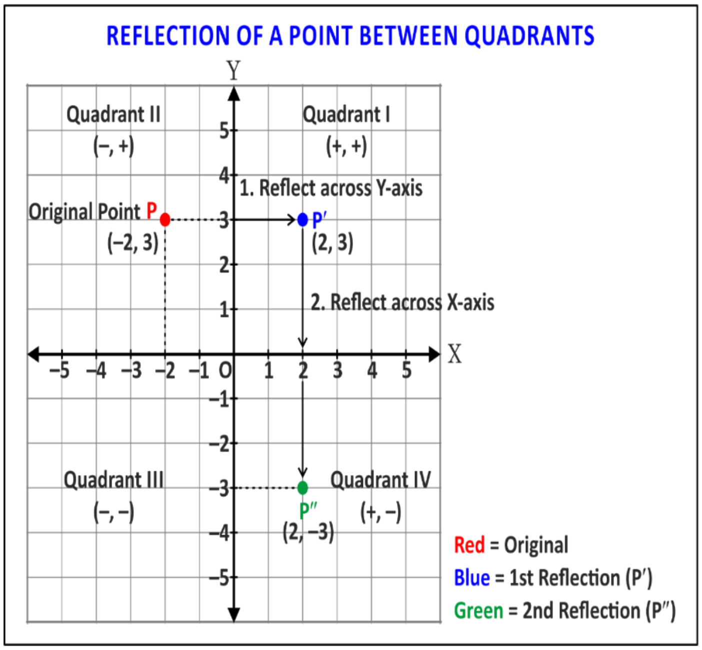
In Coordinate Geometry, points don't have to stay still! We can move them using specific rules. One of the most interesting ways to move a point is through reflection, which works exactly like a mirror.

How do mirrors work on a graph

1. **The Y-axis work as a Mirror:** When you stand in front of a vertical mirror (the Y-axis), your height (y-coordinate) stays the same, but your left and right sides (x-coordinate) swap.
2. **The X-axis work as a Mirror:** Imagine standing on a clear glass floor with a mirror underneath (the X-axis). Your position on the floor (x-coordinate) stays the same, but your "top" and "bottom" (y-coordinate) flip.

Let's discuss with an example:

Example 1: Let us take a point P, positioned in Quadrant II, having coordinates P(–2, 3). Now P takes two jumps as follows:



1. **The First Jump of P Reflects across Y-axis:**

   When the point P "jumps" over the Y-axis to the other side.

   - The y coordinate stays at +3.
   - The x coordinates at –2 move to +2.
   - The New position of P i.e. P′ will be (2, 3) in I Quadrant.

2. **The Second Jump**

   (P′ reflects across x-axis):

   Now, the mirror is the horizontal floor.

   The point P'(2, 3) "jumps" across x-axis.

   - The x coordinate stays +2.
   - The y coordinate 3 moves to –3.
   - The new position of P′ i.e. P″: (2, –3) in IV Quadrant.

   Thus, P (–2, 3) lies in II Quadrant, its reflection across y-axis P′ (2, 3) lies in I quadrant and reflection of P′ across x-axis is P″ (2, –3) lies in IV Quadrant.

The "Golden Points" to remember:

- For Reflection across Y-axis: Change the sign of x-coordinate.
- For Reflection across X-axis: Change the sign of y-coordinate.
- For Reflection across Origin: Change the sign of both x-coordinate and y-coordinate.

Why we learn this:

This is the logic used in Computer Graphics and Animation. When a character in a video game turns around or looks into water, the computer is simply performing these coordinate reflections in real-time!

EXERCISE 4.1

1. A point P lies in the III quadrant. It is reflected across the y-axis to create P′ and then P′ is reflected across the x-axis to create P″. In which quadrants does P′ and P″ lie?

2. A point A (a, b) is reflected across the x-axis to become point B. Point B is exactly 8 units below point A. What was the y-coordinate of point A?

4.5 Coordinates as Perpendicular Distances

Till now, we have thought of a point (a, b) as just a position. However, absolute value of these numbers actually represents the **perpendicular distance** of that point from the two axes.

- **The x-coordinate (a):** This tells you how far the point is from the Y-axis.
- **The y-coordinate (b):** This tells you how far the point is from the X-axis.

Note: Distance is always positive, but the coordinate value can be negative depending on which direction you move (the Quadrant).

Example 2: Point M (a, b) lies in the second quadrant. Its perpendicular distance from the x-axis is 6 units, and its perpendicular distance from the y-axis is 8 units. Find the value of $\frac{b-a}{b+a}$.

Solution: As, Distance from x-axis = 6, so the absolute value of the y-coordinate is 6.

Now, Distance from y-axis = 8, so the absolute value of the x-coordinate is 8.

Since it is in the II quadrant (–, +), the coordinates must be M (–8, 6).

Therefore, $a = -8$ and $b = 6$.

Thus, $\frac{b-a}{b+a} = \frac{6-(-8)}{6-8} = \frac{14}{-2} = -7$

Example 3: Point A$(k, 3)$ is reflected in the y-axis to form point B. Point B is then translated by 4 units upwards (parallel to the y-axis) and 2 units to the right (parallel to the x-axis) to reach its final position at point C$(-5, y)$. Determine the value of $k^2 - y$.

Solution: As, Reflecting A $(k, 3)$ in the y-axis negates the x-coordinate, so coordinates of B are $(-k, 3)$.

Now, moving 4 units up adds 4 to the y-coordinate and moving 2 units right adds 2 to the x-coordinate. Thus, coordinates of C are

$$(-k + 2, 3 + 4) = (-k + 2, 7).$$

We are given that, coordinates of C are (–5, y). Therefore:

$$-k + 2 = -5 \Rightarrow k = 7 \text{ and } y = 7.$$

Therefore, $k^2 - y = 49 - 7 = 42$

EXERCISE 4.2

1. Point P (a, b) lies in IV Quadrant. Its perpendicular distance from the y-axis is 3 units greater than its perpendicular distance from the x-axis. If the product of its coordinates is –28, find the value of the expression $a^2 + b$.

2. Point Q ($k - 4$, $2k + 7$) lies strictly in II Quadrant. The perpendicular distance of Point Q from the y-axis is exactly twice its perpendicular distance from the x-axis. Find the value of k, and evaluate the expression $k^3 + 10$.

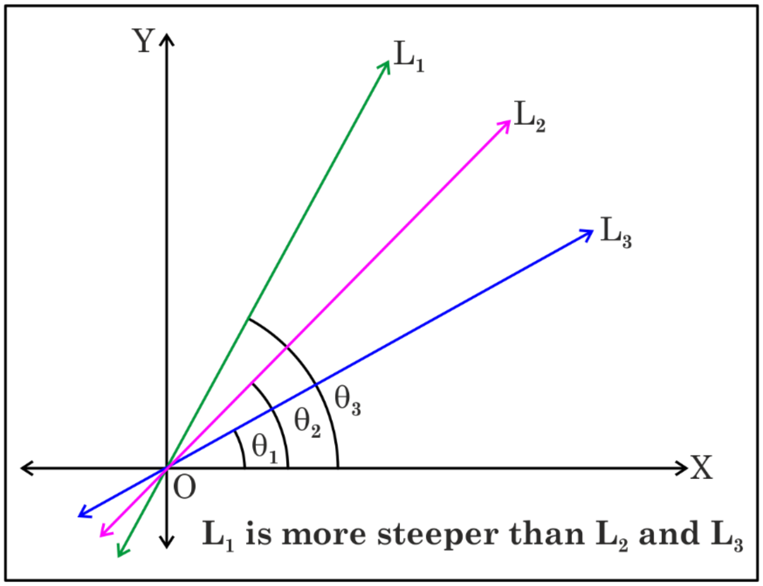
3. Let P (3, –4) be a point on the Cartesian plane. Point Q (a, b) is the image of P when reflected through y-axis. Point R(c, d) is the image of Q when reflected in the x-axis. Find the value of the $\sqrt{ac - bd}$.

4.6 The Concept of Slope (Gradient)

The slope (or gradient) denoted by m, measures the steepness and direction of a line.

- **Steepness:** A larger absolute value of m means a steeper line. A slope of 0 means the line is horizontal.

  Angle of Inclination: If a line makes an angle θ with the positive direction of the x-axis anti-clock wise, then the slope is: $m = \tan\theta$ (We will learn more about tan θ in next class)



Note: The reference line for the slope of a line is x-axis. (you will learn more about this later)

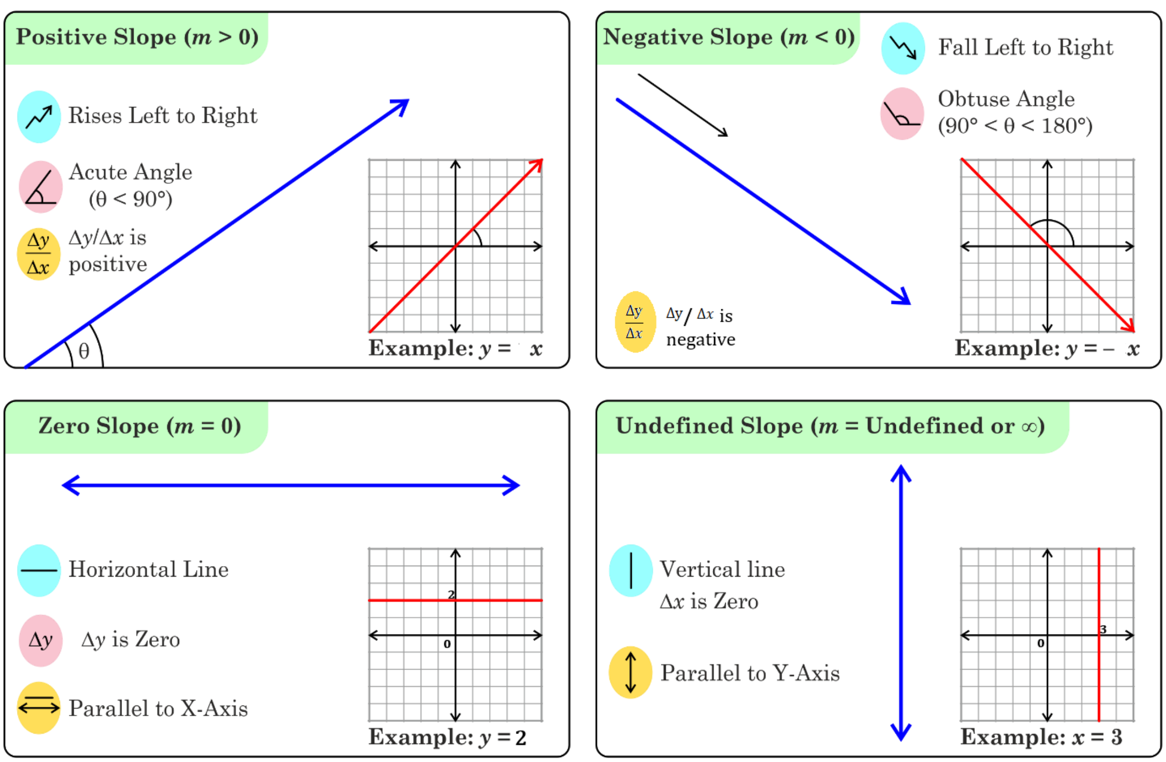
- **Direction:** A positive value of m means the line goes "uphill" from left to right. A negative m means the line goes "downhill." You can calculate it as the "rise over run" $\left(\frac{\Delta y}{\Delta x}\right)$ between any two points on the line.

  Note: Here Δ is a Greek letter read as **Delta**, and in math, it is just a shorthand for "**Difference**" or "**Change**." **Further**, Δy represents how much the height changed while Δx represents how much the horizontal distance changed. So, rise over run = $\left(\frac{\Delta y}{\Delta x}\right) = \frac{\text{Vertical Change}}{\text{Horizontal Change}}$.

  Slope of a line when two points are given: If a line passes through $A(x_1, y_1)$ and $B(x_2, y_2)$, then the slope is the "rise over run": $m = \frac{y_2 - y_1}{x_2 - x_1}$.

4.7 Properties of Slope

- **Positive Slope:** The line rises from left to right (acute angle of inclination).
- **Negative Slope:** The line falls from left to right (obtuse angle of inclination).
- **Zero Slope:** A horizontal line ($y = k$, where k is any constant).
- **Undefined Slope:** A vertical line ($x = k$, where k is any constant).



Let us define Parallel lines and Perpendicular lines using slope.

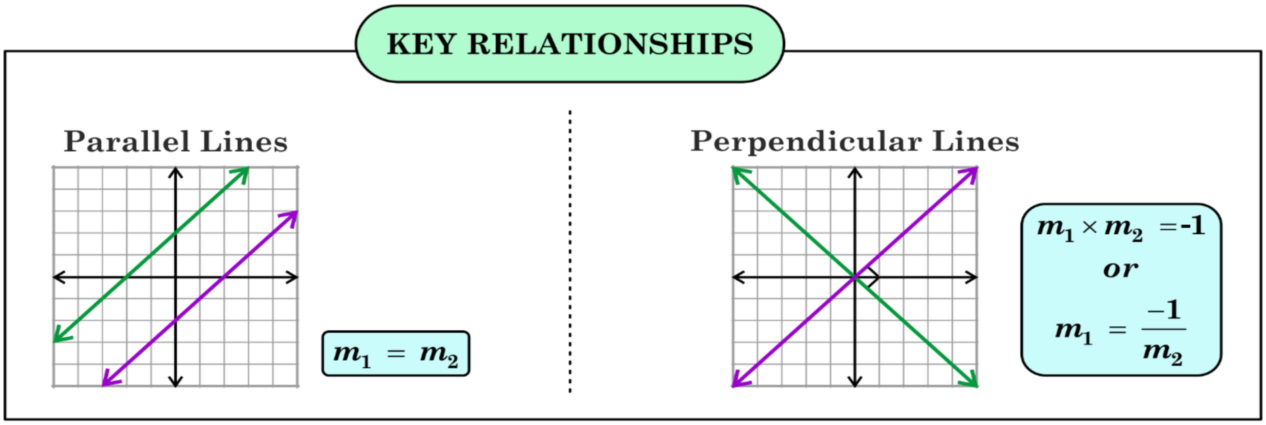
Let m_1, m_2 represents the slope of two non-vertical lines.

- **Parallel Lines:** Two non-vertical lines are parallel if and only if their slopes are equal:

$$m_1 = m_2$$

- **Perpendicular Lines:** Two non-vertical lines are perpendicular if and only if the product of their slopes is –1:

$$m_1 m_2 = -1$$



Note: In coordinate geometry, "non-vertical line" simply refers to any line that is **not** parallel to the y-axis. Further, **For a vertical line:** The change in x (Δx) is 0. Since division by zero is undefined, a vertical line has an **undefined slope**.

4.8 Intercept

The term intercept comes from the word "intersect," which means to cross or cut through. In coordinate geometry, an intercept is simply the point where a line (or any graph) crosses one of the axes on the coordinate plane.

4.8.1 The y-intercept

This is the point where a given line crosses the vertical or y-axis.

- As the y-axis cuts exactly at the zero mark on the horizontal axis, the x-coordinate is always 0 at the y-intercept.
- So, the Coordinate will be in $(0, y)$ form.
- To find the y-intercept algebraically, we simply replace x with 0 in our equation and solve for y.

4.8.2 The x-intercept

This is the point where the line crosses the horizontal x-axis.

- As the x-axis cuts exactly at the zero mark on the vertical axis, the y-coordinate is always 0 at the x-intercept.
- Coordinate Format: $(x, 0)$
- To find the x-intercept algebraically, we simply replace y with 0 in our equation and solve for x.

Example 4: The equation of a line is $y = 2x - 6$. Find x-intercept and y-intercept.

- To find the y-intercept: Put $x = 0$, we get $y = 2(0) - 6$

  So, $y = -6$. The intercept is at $(0, -6)$.
- To find the x-intercept: Put $y = 0$, we get $0 = 2x - 6$

  So, $6 = 2x \Rightarrow x = 3$. The intercept is at $(3, 0)$.

EXERCISE 4.3

1. A straight line with a slope of $\frac{1}{2}$ passes through the points $A(x, 2)$, $B(3, 4)$, and $C(7, y)$. Find the coordinates of points A and C, and calculate the value of $x + y$.
2. Points P (2, 3), Q (5, 7), and R (13, k) are three consecutive vertices of a rectangle. What is the value of k?
3. A line l passes through the origin (0, 0). It intersects the line segment joining the points A (2, 8) and B (6, 2). What is the possible range of values for the slope 'm' of line l?
4. Line l_1 passes through the points $A(1, k)$ and $B(k, 7)$. Line l_2 passes through the points $C(-2, 4)$ and $D(1, 10)$. If line l_1 is parallel to line l_2, find the value of k, and evaluate the expression $k^2 + 5$.
5. A straight line p has a slope 2. Another straight line q passes through the points M $(3a, 4)$ and N $(a, 5)$. The line p and line q intersect at a right angle. Calculate the value of a, and find the coordinates of points M and N.

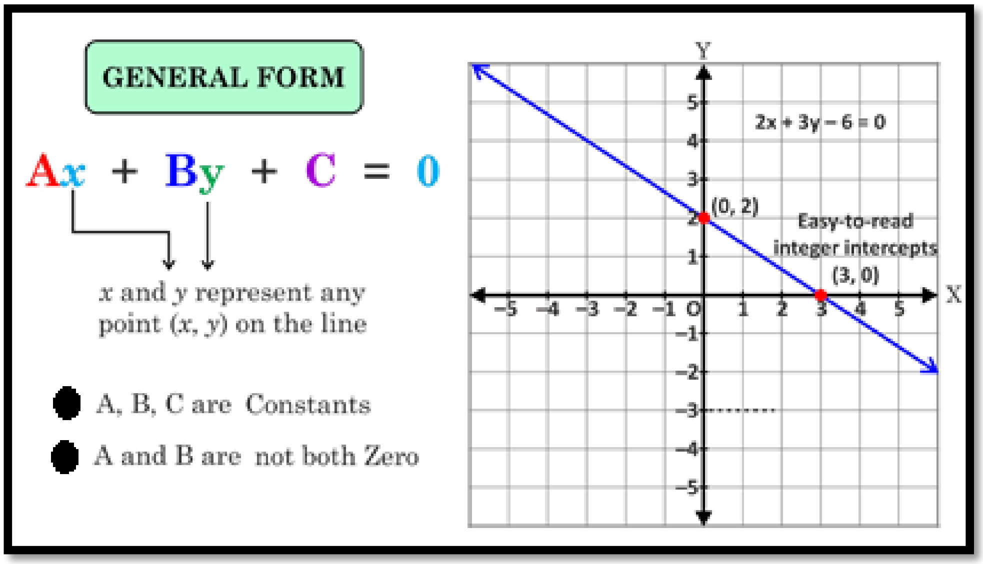
4.9 Different Forms of the Equation of a Line

A straight line can be represented algebraically in several ways depending on the given information.

4.9.1 General Form: Ax + By + C = 0

The standard algebraic representation of a straight line, where A, B, and C are constants, and A and B are non-zero at the same time.

Here, Slope, $m = \frac{-A}{B}$, x-intercept = $\frac{-C}{A}$, y-intercept = $\frac{-C}{B}$



Example 5: A straight line is given by the general equation Ax + By + C = 0. It is given that A > 0, B < 0, and C > 0. Determine through which quadrants this line will not pass through, and justify your answer using the slope and intercepts.

Solution: Let us analyze slope first. Slope, $m = \frac{-A}{B}$. Since A is positive and B is negative, so, $\frac{A}{B}$ is negative. Therefore, $m = \frac{-A}{B}$ becomes positive. The line goes uphill.

Now, y-intercept = $\frac{-C}{B}$. Since C is positive and B is negative. Therefore, ***y*-intercept** becomes positive. The line crosses the upper half of the y-axis.

Further, x-intercept = $\frac{-C}{A}$. Since C is positive and A is positive. Therefore, ***x*-intercept** becomes negative. The line crosses the left half of the x-axis.

A line that crosses the negative x-axis, the positive y-axis, and has a positive upward slope must sweep through Quadrants I, II, and III. **It never enters Quadrant IV**.

Example 6: The coefficients of the line $Ax + By + C = 0$ are not random and are such that $2B = A + C$. Prove that no matter what specific numbers you choose for A, B, and C (as long as they follow this rule), the line will always pass through one specific, fixed point. Find that point.

Solution: Let's rewrite the given condition $2B = A + C$ as $A - 2B + C = 0$

Now on comparing this equation with the given equation of line

$$Ax + By + C = 0$$

We get $x = 1$ and $y = -2$.

Therefore, the equation is always satisfied when $x = 1$ and $y = -2$.

Thus, we can say that line always pass through one specific fixed point and the fixed point is $(1, -2)$.

EXERCISE 4.4

1. The line $kx + 3y - 12 = 0$ forms a right-angled triangle with the x and y coordinate axes. If the total area of this triangle is 12 square units, find all possible values for the slope of this line.

2. The straight line $px + qy + r = 0$ (where $p, q, r \neq 0$) forms an isosceles right-angled triangle with the coordinate axes in the first quadrant. What must be the algebraic relationship between the coefficients p and q?

3. Line l_1 has the equation $3x - 5y + 10 = 0$.

   Line l_2 has the equation $5x + 3y + K = 0$.

   Prove algebraically that l_1 and l_2 are perpendicular.

   Also, if the x-intercept of l_1 is identical to the y-intercept of l_2, then find the value of K.

4. A line is written in the general form $kx - y + C = 0$. You are given two clues about this line:

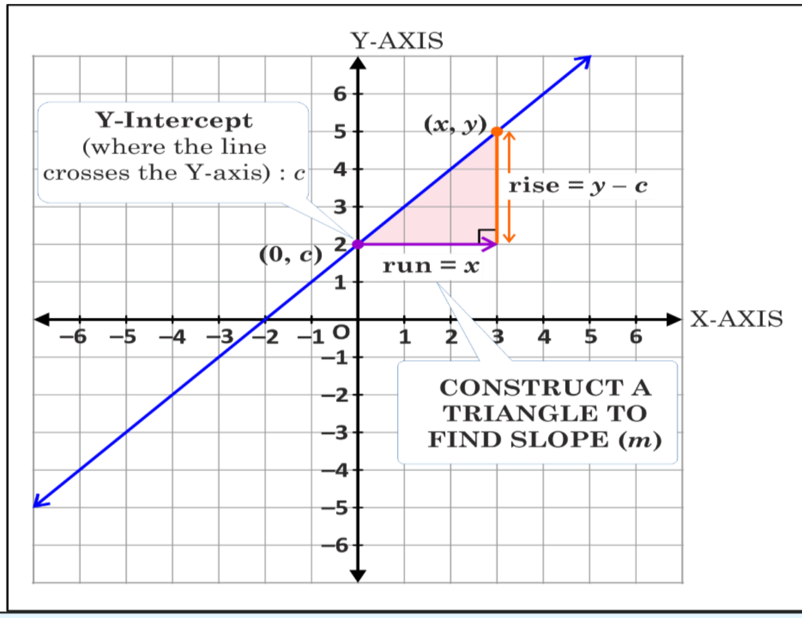
   Clue 1: The line passes through the coordinate point (3, 10).

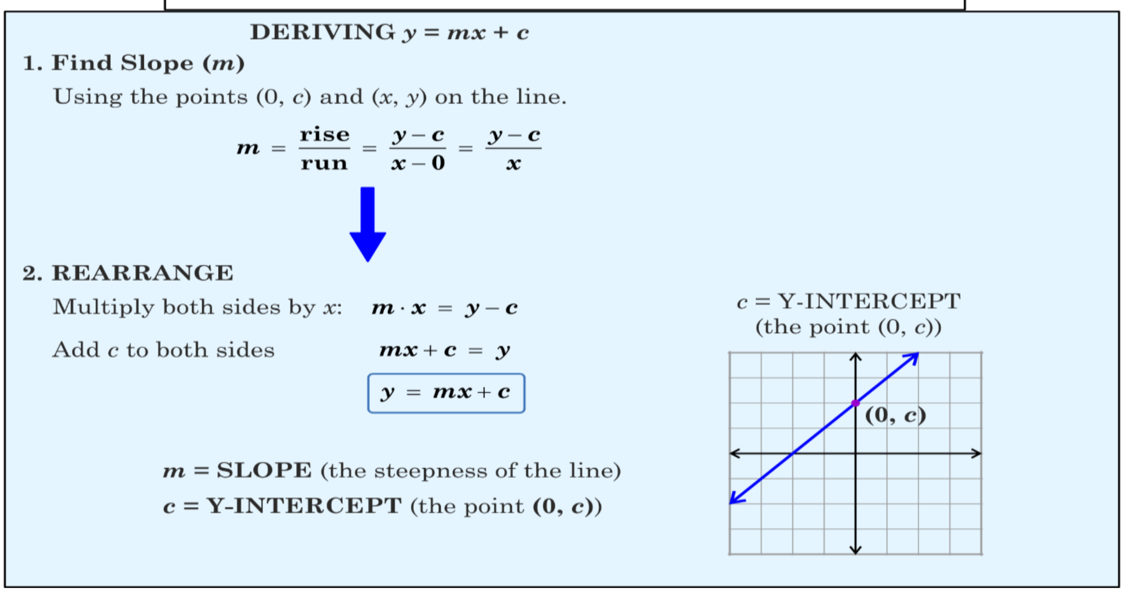
   Clue 2: The sum of its x-intercept and y-intercept is exactly equal to its slope.

   Find all possible equations for this line.

4.9.2 Slope-Intercept form: $y = mx + c$

Slope-Intercept Form is used when we know the slope 'm' and the y-intercept 'c' (y-intercept means where the line crosses y-axis)





Example 7: Find the equation of the line in slope-intercept form that has a slope of $\frac{1}{2}$ and passes through the point (0, –2).

Solution: Given, slope of the line, $m = \frac{1}{2}$ and since line passes through the point (0, –2) means y-intercept, $c = -2$

Thus, required equation of the line is $y = \frac{1}{2}x - 2$.

Example 8: A straight line passes through the points (–3, 5) and (1, –3). Determine the equation of this line in slope-intercept form.

Solution: Let the equation of the required line is $y = mx + c$, so $m = \frac{-3-5}{1+3} = -2$

Thus, equation line becomes $y = -2x + c$

Since the line is passes through (–3, 5), so $5 = 6 + c \Rightarrow c = -1$

Therefore, required equation of the line is $y = -2x - 1$.

Example 9: Convert the linear equation $3x - 4y + 12 = 0$ into slope-intercept form.

Solution: The given equation $3x - 4y + 12 = 0$ can be written as $4y = 3x + 12$

Thus, $y = \frac{3}{4}x + 3$ is the required slope-intercept form.

EXERCISE 4.5

1. A straight line crosses the y-axis at the point (0, –5). From that point, it rises 3 units vertically for every 2 units it moves horizontally to the right. Write the equation of this line in slope-intercept form.

2. Write the equation of a line in slope-intercept form that is parallel to the line $y = 5x - 12$ and passes through the point (0, 9).

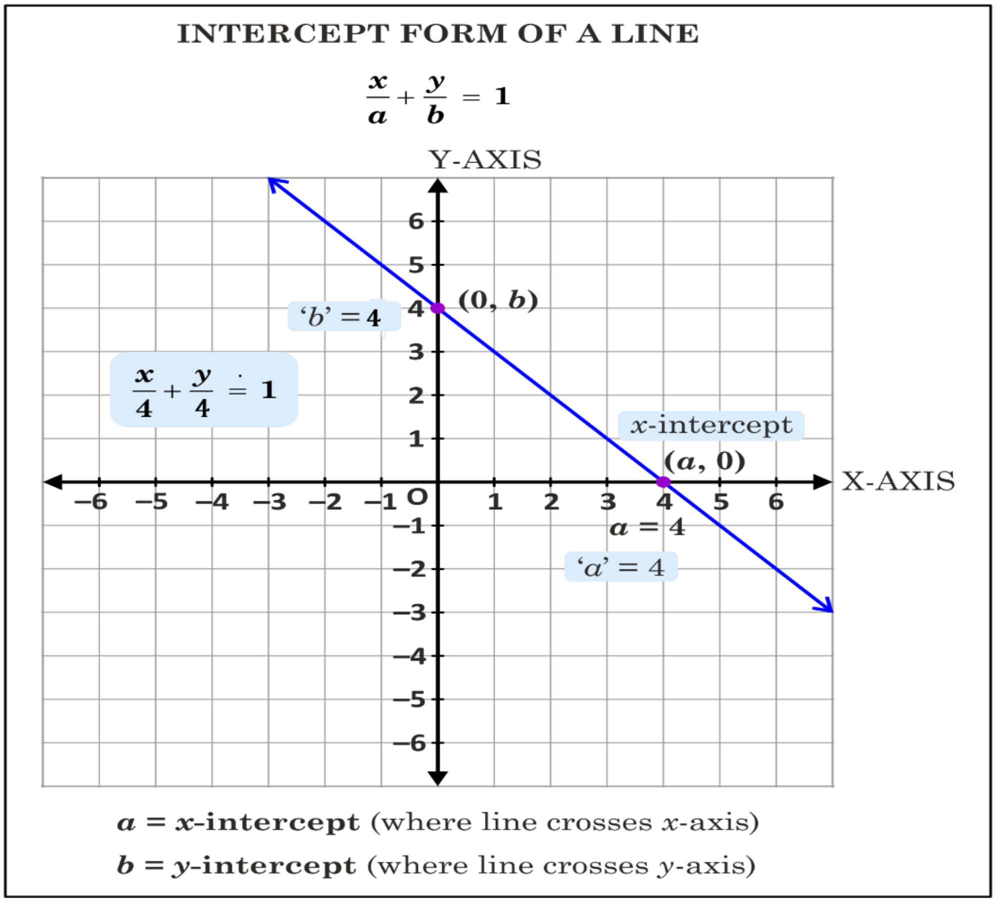
3. Determine the equation of the line in slope-intercept form that is perpendicular to the line $y = \frac{1}{3}x + 4$ and passes through the origin.

4. A straight line passes through the points (0, –6) and (4,10). Find the equation of this line in slope-intercept form.

5. Convert the linear equation $8x - y + 7 = 0$ into slope-intercept form.

4.9.3 Intercept Form: $\frac{x}{a}+\frac{y}{b}=1$

Intercept Form is used when we know both the x-intercept (a) and the y-intercept (b)



Example 10: A straight line passes through the point (–4, 5). The line makes intercepts on both the coordinate axes that are equal in magnitude, but opposite in sign. Find the equation of the line.

Solution: Let the equation of required line be $\frac{x}{a}+\frac{y}{b}=1$

It is given that intercepts are equal in magnitude, opposite in sign means if the x-intercept is a, then y-intercept must be $-a$. (So, $b=-a$).

Now, equation of the line is $\frac{x}{a}+\frac{y}{-a}=1 \Rightarrow x-y=a$

Since, line passes through (–4, 5) so this point must satisfies the equation of line, $-4-5=a \Rightarrow a=-9$

Therefore, required equation of line is $x-y=-9$.

Example 11: A moving straight line intercepts x-axis at a and y-axis at b. As the line moves, its intercepts constantly change, but they *always* satisfy the mathematical relation: $\frac{2}{a}+\frac{3}{b} = 5$

Prove that this moving line will always pass through one specific fixed point. Also, find the coordinates of that fixed point.

Solution: Let the equation of required line be $\frac{x}{a}+\frac{y}{b} = 1$

It is given that,

$$\frac{2}{a}+\frac{3}{b} = 5$$

$$\Rightarrow \quad \frac{2/5}{a}+\frac{3/5}{b} = 1$$

On comparing the two equations, we get $x = \frac{2}{5}, y = \frac{3}{5}$

Thus, the standard equation is *always* satisfied if $x = \frac{2}{5}, y = \frac{3}{5}$, regardless of what a and b are. Therefore, fixed point is $\left(\frac{2}{5}, \frac{3}{5}\right)$.

EXERCISE 4.6

1. A straight line passes through the point (2, 3) and forms a right-angled triangle with the positive x and y axes. If the area of this triangle is 12 square units, find the values of its x-intercept (a) and y-intercept (b).

2. A straight line passes through the point (2, 2). The sum of its x-intercept (a) and its y-intercept (b) is 9. Determine the value of the product of its intercepts.

3. A straight line passes through the point (3, 5). The sum of its x-intercept and its y-intercept is zero. Find the equation(s) of all possible lines that satisfy these conditions.

4. A straight line forms a right-angled triangle with the positive x and y-axes. The total area of this triangle is 24 square units, and the length of its hypotenuse (the line segment intercepted between the axes) is 10 units. Find all possible equations of this line in the intercept form.

5. A straight line passes through the point (3, 2). The x-intercept (a) and y-intercept (b) of this line are both positive numbers. If the sum of its intercepts is 12, find all possible equations for this line.

Summary

1. **The Cartesian System**
   - **Definition:** Coordinate geometry bridges algebra and geometry by using a coordinate system to study shapes and positions.
   - **The Plane:** It is formed by the intersection of a horizontal x-axis and a vertical y-axis at a central point called the origin (0, 0).
   - **Quadrants:** The axes divide the plane into four regions:
     - **Quadrant I:** Positive x, Positive y (+, +).
     - **Quadrant II:** Negative x, Positive y (–, +).
     - **Quadrant III:** Negative x, Negative y (–, –).
     - **Quadrant IV:** Positive x, Negative y (+, –).
   - **Point Representation:** Every point P is an ordered pair (x, y), where x is the abscissa (horizontal distance) and y is the ordinate (vertical distance).

2. **The Concept of Slope (*m*)**
   - **Definition:** Slope measures the steepness and direction of a line with reference to x-axis.

     Calculation: For a line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$, the slope is calculated as the

     "rise over run": then the slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
   - **Direction:**
     - **Positive Slope ($m > 0$):** Line rises from left to right (acute angle).
     - **Negative Slope ($m < 0$):** Line falls from left to right (obtuse angle).
     - **Zero Slope ($m = 0$):** The line is horizontal (parallel to x-axis).
     - **Undefined Slope:** The line is vertical (parallel to y-axis).

3. **Key Line Relationships**
   - **Parallel Lines:** Two lines are parallel if their slopes are equal $(m_1 = m_2)$.
   - **Perpendicular Lines:** Two lines are perpendicular if the product of their slopes is –1

$$(m_1 \times m_2 = -1).$$

4. **Intercepts**

- **Intercept:** The exact point where a line crosses an axis.
- ***y*-intercept:** Where the line crosses the y-axis; the x-coordinate is always 0, denoted as $(0, y)$.
- ***x*-intercept:** Where the line crosses the x-axis; the y-coordinate is always 0, denoted as $(x, 0)$.

5. **Forms of the Equation of a Line**

A straight line can be written in three primary algebraic forms:

- **General Form:** $Ax + By + C = 0$, where A, B, C are constants.
- **Slope-Intercept Form:** $y = mx + c$, where m is the slope and c is the y-intercept.
- **Intercept Form:** $\frac{x}{a} + \frac{y}{b} = 1$, where a is the x-intercept and b is the y-intercept.

Chapter 5

Combinatorics

5.1 Introduction: The Mathematics of Possibilities

Imagine you are setting a 4-digit passcode for your tablet. You can use any digit from 0 to 9. How many different passcodes can you create if you do not repeat any digit? If you try to list them all — 0123, 0124, 0125…, it may take lot of time, and you may miss a few of them. Here is a branch of mathematics that can help you to find out all the possible combinations of different passcodes, without missing any. This unit will talk about combinatorics, a branch of mathematics that gives us powerful tools to find the total number of arrangements, and selections quickly and accurately, without listing them. It forms the backbone of modern computer science, probability theory, and data security.

Combinatorics is an important part of everyday decision making: how many passwords can you create? How many ways can a teacher arrange students on a bench? How many different cricket teams can be formed from a squad? By the end of this chapter, you will be able to answer all of these questions easily.

Historical Note: The Evolution of Combinatorics

Combinatorics, the study of counting and arrangements, has roots in ancient mathematics. In India, ideas of combinations appeared in concepts like *vikalpa* (choices and arrangements). The *Chandaḥśāstra* of Pingala (c. 3rd – 2nd century BCE) studied patterns in poetic meters and used methods equivalent to binary representation.

Ancient texts like the *Sushruta Samhita* described six basic tastes and their mixtures, which correspond to 63 possible combinations. Later, mathematicians such as *Mahavira* (9th century) in *Gaṇita Sāra Saṃgraha* and *Bhāskara* II (12th century) in *Līlāvatī* developed more systematic methods for counting arrangements.

In Europe, combinatorics became more structured through the work of mathematicians like Pascal and Jacob Bernoulli, whose book *Ars Conjectandi* (1713) helped establish the foundations of modern combinatorics and probability.

5.2 The Fundamental Principle of Counting (FPC)

To understand combinatorics, let us first understand the fundamental principle of counting.

Every complex counting problem can be broken down into smaller, individual, step-by-step decisions. The Fundamental Principle of Counting provides two core rules to calculate the total number of possible outcomes.

The Rule of Multiplication (The "AND" Rule):

If one event can happen in m different ways, AND a second independent event can happen in n different ways, then the total number of ways both events can happen together is $m \times n$. We use this rule when an event requires multiple steps to be completed together.

The Rule of Addition (The "OR" Rule):

If an event can be completed by choosing from one set of m options OR a distinct, non-overlapping set of n options, then the total number of ways to complete the event is $m + n$. We use this rule when we have mutually exclusive (i.e., disjoint) choices.

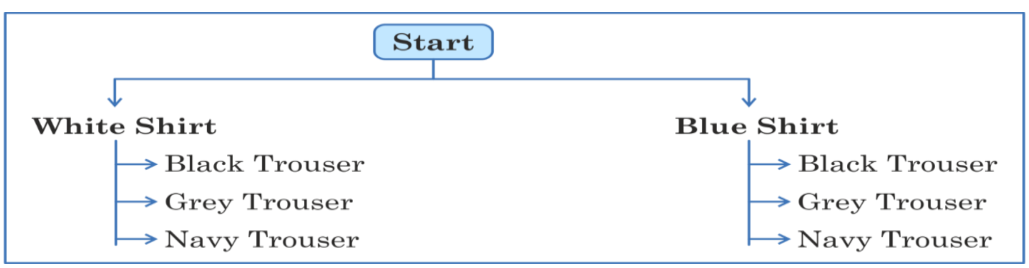
Let us understand these rules using a visual *Tree Diagram*.

5.2.1 Visualizing the Principle: The Tree Diagram

A tree diagram, as the name suggests, maps out every possible outcome by branching out at each decision point. It is especially useful for small problems to verify your multiplication.

Suppose a student is choosing an outfit. He has 2 types of shirts (White, Blue) and 3 types of trousers (Black, Grey, Navy). In how many different ways, the student can choose his outfits? Here, the choices of shirt and choices of trousers are two independent events. Choosing a complete outfit will require a combination of shirt and trouser.

Let us understand this by drawing the following Tree Diagram:



By counting the endpoints of the tree, we see there are exactly 6 unique outfit possibilities. Instead of drawing a big tree for larger problems, we use the Fundamental Principle of Counting to simply multiply the number of branches at each step: 2 (shirts) × 3 (trousers) = 6 outfits.

Let us understand the fundamental principle of counting with some more examples.

Example 1: In a class there are 12 boys and 10 girls. The teacher wants to select 1 boy and 1 girl to represent the class for a function. In how many ways can the teacher make the selection?

Solution: Here the teacher is to perform two operations:

(i) selecting 1 boy out of 12 boys, and

(ii) selecting 1 girl out of 10 girls.

The first of these can be done in 12 ways and the second in 10 ways.

So, by the fundamental principle of multiplication, the required number of ways = $12 \times 10 = 120$.

Hence, the teacher can make the selection of 1 boy and 1 girl in 120 ways.

Example 2: A customer is choosing specifications for a new laptop online. The customer must choose 1 processor from 3 options, 1 screen size from 2 options, and 1 color from 4 options. How many unique laptop configurations are possible?

Solution: The customer is making three independent choices that must all happen together to build the laptop. The processor can be chosen in 3 ways, the screen size in 2 ways, and the color in 4 ways. By Fundamental Principle of Counting, the total number of unique configurations is $3 \times 2 \times 4 = 24$.

Example 3: A quiz has 5 multiple-choice questions. Each question has 4 options A, B, C, and D. In how many different ways can a student answer the entire quiz?

Solution: The student must answer question 1, question 2, and so on, up to question 5. Each of the 5 questions provides exactly 4 choices. By the fundamental principle of counting, the total number of ways to answer the quiz is $4 \times 4 \times 4 \times 4 \times 4$, which equals 4^5 or 1024 ways.

Example 4: A school offers 4 sports clubs, 3 music clubs, and 2 debate clubs. A student wants to join exactly one activity. In how many ways can the student make a choice?

Solution: The student joins a sports club OR a music club OR a debate club — exactly one, so the groups are mutually exclusive i.e., disjoint (OR rule).

$\therefore$ Total number of ways = $4 + 3 + 2 = 9$

Example 5: How many 2-digit numbers can be formed using the digits 1, 2, 3, 4, 5 if repetition of digits is not allowed?

Solution: A 2-digit number has a tens place and a unit's place.

The ten's place can be filled in 5 ways and the unit's place can be filled with the remaining 4 digits (as the repetition is not allowed).

$\therefore$ The total number of possible 2-digit numbers = $5 \times 4 = 20$

EXERCISE 5.1

1. A restaurant offers 4 starters, 5 main courses, and 3 desserts. In how many ways can a 3-course meal be ordered?

2. There are 5 doors to enter a hall and 3 different doors to exit. In how many ways can a person enter and exit the hall?

3. A bicycle lock has 3 dials, each with digits 0 to 9. How many different lock combinations are possible if a digit can be repeated?

4. How many numbers between 2000 and 3000 can be formed from the digits 2, 3, 4, 5, 6, 7 when repetition of digits is not allowed?

5. How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 6 without repetition? What if the repetition of digits is allowed?

6. How many numbers are there between 100 and 1000 such that 9 is in the unit's place? How many numbers will be there if 9 is at the ten's place? How this number will change, if 9 is at the hundred's place. Do you see a pattern? Can you describe this in your language?

5.3 Factorials

When solving counting problems, we frequently encounter the multiplication of consecutive descending natural numbers. For instance, if you want to arrange 4 distinct books on a display shelf, the Fundamental Principle of Counting tells us there are 4 choices for the first spot, 3 choices for the second spot, 2 for the third, and 1 for the fourth. The total number of arrangements is $4 \times 3 \times 2 \times 1 = 24$.

As the number of objects increases, writing these long multiplication sequences becomes incredibly tedious. To express this concept efficiently, mathematicians have introduced a special mathematical operator called the **Factorial**.

The product of the first n natural numbers is called "n factorial" and is denoted by symbol $n!$.

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

For example: $3! = 3 \times 2 \times 1 = 6,$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

We take $0! = 1$ (Can you guess why?)

Example 6: Evaluate 6!

Solution: By definition,

$$\begin{aligned} 6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720 \end{aligned}$$

Example 7: Evaluate $\frac{8!}{5!}$

Solution:

$$\begin{aligned} \frac{8!}{5!} &= \frac{8 \times 7 \times 6 \times 5!}{5!} \\ &= 8 \times 7 \times 6 \\ &= 336 \end{aligned}$$

The Expansion Rule: You can stop expanding a factorial at any stage by placing the factorial sign on the last number. For example, $6! = 6 \times 5!$ $= 6 \times 5 \times 4!$, ... and so on.

Example 8: Is the statement $3! + 4! = 7!$ true or false? Verify.

Solution:

$$\begin{aligned} \text{LHS} &= 3! + 4! \\ &= 6 + 24 = 30 \\ \text{RHS} &= 7! = 5040 \\ 30 &\neq 5040 \end{aligned}$$

The statement is FALSE.

Remember: $a! + b! \neq (a + b)!$; $a! - b! \neq (a - b)!$

Example 9: Find n, if $\frac{n!}{(n-2)!} = 42$

Solution:

$$\frac{n!}{(n-2)!} = 42$$

$$\Rightarrow \frac{n \times (n-1) \times (n-2)!}{(n-2)!} = 42$$

$$\Rightarrow n(n-1) = 42$$

$$\Rightarrow n = 7 \ (\because 7 \times 6 = 42)$$

Use hit-and trial method.

Example 10: Find x, if $\frac{1}{4!}+\frac{1}{5!} = \frac{x}{6!}$.

Solution:

$$\frac{1}{4!}+\frac{1}{5!} = \frac{x}{6!}$$

$$\Rightarrow \quad x = \frac{6!}{4!}+\frac{6!}{5!}$$

$$\Rightarrow \quad x = (6 \times 5) + 6$$

$$\Rightarrow \quad x = 36$$

Example 11: Find the value of $\frac{10!}{9!+8!}+\frac{9!}{8!+7!}+\frac{8!}{7!+6!}+\ldots+\frac{3!}{2!+1!}$.

Solution: Here, we observe that each term can be written as $\frac{(x+2)!}{(x+1)!+x!}$, where $x = 1, 2, \ldots, 8$

Further, $$\frac{(x+2)!}{(x+1)!+x!} = \frac{(x+2)(x+1)}{(x+1)+1} = x+1$$

Thus, $$\frac{10!}{9!+8!}+\frac{9!}{8!+7!}+\frac{8!}{7!+6!}+\ldots+\frac{3!}{2!+1!} = 9 + 8 + \ldots + 2 = 44$$

Example 12: Find the value of $\sqrt{x-y}$, if

$$1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \cdots + 2025 \times 2025! = x! - y!$$

Solution: Observe that
$$2 \times 2! = (3 - 1) \times 2! = 3 \times 2! - 1 \times 2! = 3! - 2!$$

Similarly, each term of the given expression can be written as difference of two factorials,

So, $S = 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \ldots + 2025 \times 2025!$

$S = (2! - 1!) + (3! - 2!) + (4! - 3!) + \ldots + (2026! - 2025!)$

$S = 2026! - 1!$

Thus, $x = 2026$, $y = 1$

$$\therefore \quad \sqrt{x-y} = \sqrt{2025} = 45$$

Exercise 5.2

1. Find the HCF and LCM of 6! , 5!.
2. Find n if $\frac{n!}{(n-3)!} = 60$.
3. Find x, if $\frac{1}{5!}+\frac{1}{6!} = \frac{x}{7!}$.
4. Find the value(s) of x in each of the following: (Here, $x \geq 0$)
   (a) $x! = 120$
   (b) $(x!)^2 = 576$
   (c) $(x!)^2 - 25x! + 24 = 0$
   (d) $\frac{(x+2)!-(x+1)!}{x!} = 49$

5.4 Permutations: When Order Matters

Suppose three students — Aarav (A), Bhavna (B), and Chirag (C), are to be seated in a row of 3 chairs. The seating ABC is entirely different from BCA. Here, the order of arrangement matters. Such arrangements are called Permutations.

When we arrange **distinct** objects in a specific sequence, the **order matters**. Arranging the letters, A, B, and C as "CAB" is fundamentally different from "BAC". We use Permutations when dealing with situations where order is important, such as creating schedules, seating people in chairs, generating passwords, and awarding rankings or roles.

Let's learn that how does Permutation's work.

5.4.1 The Permutation Formula

The number of permutations of n distinct objects taken r at a time is given by: ${}^{n}P_{r} = \frac{n!}{(n-r)!}$; $0 \leq r \leq n$, where no repetition is allowed.

The number of permutations of n different objects taken r at a time, where repetition is allowed, is n^r.

The number of permutations of n distinct objects taken all n at a time is given by ${}^{n}P_{n} = n!$

Example 13: How many 4-letter words (with or without meaning) can be formed using the letters of the word PENCIL, if no letter is repeated?

Solution: PENCIL has 6 distinct letters. We arrange 4 of them in order in ${}^{6}P_{4}$ ways i.e., $\frac{6!}{(6-4)!} = \frac{6!}{2!} = 6\times5\times4\times3 = 360$ ways.

Example 14: In an 8-person race, how many ways can Gold, Silver, and Bronze be awarded?

Solution: We arrange 3 runners into specific medal positions from the 8 persons. The order matters here because winning Gold is different from winning Bronze.

The required number of ways is $^8P_3 = \frac{8!}{5!} = 8\times7\times6 = 336$ ways.

Example 15: In how many ways can a 12-member club elect a President, Vice-President, and a secretary if each member can hold only one post?

Solution: We select and arrange 3 members for distinct roles. Since the roles are different, the order of selection matters. So, required number of required ways are $^{12}P_3 = \frac{12!}{9!} = 12\times11\times10 = 1320$ ways.

Example 16: In how many ways can 6 students stand in a straight line for a photograph?

Solution: All 6 students are being arranged in all 6 positions in $^6P_6 = 6! = 720$ ways.

Example 17: How many 4-digit numbers greater than 7000 can be formed using the digits 2, 3, 5, 7, 8, and 9 if the repetition of digits is not allowed?

Solution: For the number to be greater than 7000, the thousands digit must be 7, 8, or 9 giving 3 choices.

After fixing the thousands digit, we arrange the remaining 3 places using 5 remaining digits in $^5P_3 = \frac{5!}{2!} = 5\times4\times3 = 60$ ways.

$\therefore$ By the Fundamental Principle of Multiplication, the required total numbers formed are $3 \times 60 = 180$.

Think: What if the digits can be repeated?

Example 18: A library shelf has space for 7 books. There are 4 Mathematics books and 3 Science books. In how many ways can they be arranged if all 3 Science books must always be kept together?

Solution: Let us consider the 3 Science books as one single block. Now we have, 4 Math's books and 1 block of science books. So, we have to arrange total 5 units.

Now, the 5 units can be arranged in a row in 5! ways i.e., 120 ways.

Now, the 3 Science books within their block can themselves be arranged in 3! ways i.e., 6 ways.

$\therefore$ By the Fundamental Principle of Multiplication, the total number of ways are 120×6 i.e., 720 ways.

Exercise 5.3

1. In how many ways can 4 distinct cars be parked in 6 empty spaces?
2. How many 3-letter words (with or without meaning) can be formed using the letters of the word "LOGIC"?
3. A library has 5 distinct science books and 3 distinct math books. In how many ways can they be arranged on a shelf if the math books must occupy the first 3 positions?
4. In how many ways can 5 boys and 2 girls be seated in a row of 7 chairs if the 2 girls must always sit together?
5. How many 4-digit numbers can be formed using the digits 2, 4, 6, 8, 9 (without repetition) if the number must be strictly greater than 6000?
6. How many words (with or without meaning) can be formed from the letters of the word, 'DAUGHTER', so that:

   (i) all vowels occur together?

   (ii) all vowels do not occur together?

5.5 Combinations: When Order Does NOT Matter

In the previous section, we learned that when arranging objects (like assigning a President, Vice-President, and Secretary), where the order is important. But what happens when the order does not matter at all?

Suppose a teacher needs to select a team of 3 students from a group of 10 to represent the class. If the teacher selects Amit, Rahul, and Seema, they form a specific group. If the teacher selects Seema, Amit, and Rahul, it is the exactly the same group of students representing the class. The 3 students constitute a team so that the arrangement of students is irrelevant.

Whenever we are dealing with selections, groupings, or collections where sequence is irrelevant, we use **Combinations**. We apply combinations when forming committees, dealing hands of cards, joining geometric points, or choosing toppings for a pizza.

5.5.1 The Combination Formula

The number of combinations (selections) of n distinct objects taken r at a time is given by:

$$^{n}C_{r} = \frac{n!}{r!(n-r)!};\ 0 \le r \le n$$

Some useful results:

$$^nC_0 = {^nC_n} = 1$$

$$^nC_1 = {^nC_{n-1}} = n$$

$$^nC_r = {^nC_{n-r}}$$

$$^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$$

> **The relation between ^nP_r and ^nC_r is given by $^nP_r = r! \times {^nC_r}$**

Note: You will explore these formulas in detail in higher classes.

Example 19: In how many ways can a committee of 4 members be chosen from a group of 9 people?

Solution: Since the order of selection of members does not matter in the formation of a committee, this is a problem of combinations.

The number of ways of selecting 4 persons out of 9 is given by $^9C_4 = \dfrac{9!}{4! \times 5!} = \dfrac{9\times8\times7\times6}{4\times3\times2\times1} = 126$ ways.

Example 20: A student must answer any 6 questions out of 9 in an examination. In how many ways can the student make the selection?

Solution: Since the order of selection of questions does not matter, this is a problem of combinations.

The number of ways of selecting 6 questions out of 9 is given by $^9C_6 = \dfrac{9!}{6! \times 3!} = \dfrac{9\times8\times7}{3\times2\times1} = 84$ ways.

Example 21: From 6 boys and 4 girls, in how many ways can a team of 3 boys and 2 girls be formed?

Solution: The team consists of 3 boys and 2 girls. Since the order of selection does not matter, we calculate the ways of selecting boys and girls separately and multiply them.

Selection of 3 boys from 6 boys is done in

$$^6C_3 = \frac{6!}{3! \times 3!} = \frac{6\times5\times4}{3\times2\times1} = 20 \text{ ways.}$$

Selection of 2 girls from 4 girls is done in $^4C_2 = \dfrac{4!}{2! \times 2!} = \dfrac{4\times3}{2\times1} = 6$ ways.

Now, the total number of ways to form the team by applying the fundamental principle of counting

= (ways of selecting 3 boys) × (ways of selecting 2 girls) = 20 × 6 = 120

Hence, the required number of ways is 120.

Example 22: There are 8 non-collinear points in a plane. How many straight lines can be drawn amongst them?

Solution: Since any two points determine exactly one straight line and the 8 points are non-collinear (no three points lie on the same line), each pair of points will form a unique straight line.

To find the total number of straight lines, we need to find the number of ways of selecting 2 points out of 8 points, which is given by $^8C_2 = \frac{8!}{2! \times 6!} = \frac{8 \times 7}{2} = 28$. Hence, the total number of straight lines that can be drawn is 28.

Exercise 5.4

1. In how many ways can 3 students be chosen from a class of 12 to represent the school?
2. How many triangles can be formed from 12 points in a plane, of which 5 are collinear?
3. An examination paper contains 12 questions divided into two parts, A and B. Part A contains 7 questions and Part B contains 5 questions. A candidate is required to attempt 7 questions, selecting at least 3 from each part. In how many ways can the candidate select the questions?
4. How many diagonals does a polygon with 10 sides have?
5. If you invite 15 of your friends to a party and all shake hands exactly once, how many handshakes occur?
6. A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

5.6 Combinatorics in the Modern World

Cybersecurity: Password Safety

When someone tries to access an account without permission, they may use a computer program that quickly tries many different password combinations until it finds the correct one.

If a website allows an 8-character password using only lowercase letters (26 choices), the total number of possible passwords is $26^8 \approx 208$ billion. Modern computers can test millions of passwords in a short time. That is why websites ask users to include uppercase letters, numbers, and symbols in their passwords.

When more types of characters are allowed (for example, about 70 choices instead of 26), the number of possible passwords increases greatly to $70^8 \approx 576$ trillion. This large increase makes it much harder for a computer program to guess the correct password quickly.

Data Processing and Task Scheduling

In cloud computing and AI systems, servers often receive multiple tasks at the same time. The system must decide the order in which these tasks should be processed efficiently.

If a server receives 8 different tasks at the same time, it can arrange them in different orders to process them one by one. The number of such possible orders is $8! = 40{,}320$. This means there are 40,320 different ways to arrange the tasks.

By studying such arrangements, software engineers design systems that choose efficient processing orders, saving time and energy.

Summary

- The Fundamental Principle of Counting (FPC):
    - **The Rule of Multiplication (The "AND" Rule):** If one event can happen in different ways, AND a second independent event can happen in n different ways, then the total number of ways both events can happen together is $m \times n$.
    - **The Rule of Addition (The "OR" Rule):** If a task can be completed by choosing from one set of n options OR a distinct, non-overlapping set of n options, then the total number of ways to complete the task is $m + n$.

- $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$, where n is a natural number.
- The number of permutations of n distinct objects taken r at a time is given by: ${}^{n}P_{r} = \frac{n!}{(n-r)!}; 0 \le r \le n$, where repetition is not allowed.
- The number of permutations of n different objects taken r at a time, where repetition is allowed, is n^r.
- The number of permutations of n distinct objects taken all at a time is given by ${}^{n}P_{n} = n!$.
- The number of combinations (selections) of n distinct objects taken r at a time is given by: ${}^{n}C_{r} = \frac{n!}{r!(n-r)!};\ 0 \le r \le n$
- Some useful results:

$$
\begin{aligned}
{}^{n}C_{0} &= {}^{n}C_{n} = 1 \\
{}^{n}C_{1} &= {}^{n}C_{n-1} = n \\
{}^{n}C_{r} &= {}^{n}C_{n-r} \\
{}^{n}C_{r} + {}^{n}C_{r-1} &= {}^{n+1}C_{r}
\end{aligned}
$$

Permutation vs Combination

Let us have a quick check to revisit the key difference between Permutation and Combination. Let there be three letters: A, B, C and we take 2 of them.

Feature	Permutation	Combination
Core Idea	***Arranging*** the chosen letters	Just ***selecting*** the letters
Does order matter?	Yes. AB and BA are different.	No. AB and BA are the same.
Process	First choose and then arrange them.	Just choose the pair.
Possible results	AB, BA, AC, CA, BC, CB	AB, AC, BC
Total cases	${}^{3}P_{2} = 6$	${}^{3}C_{2} = 3$

Chapter 6
Exploring Some More Progressions

6.1 Introduction

Let us observe a few examples of Progressions below:

$$4, 8, 16, 32, \ldots$$

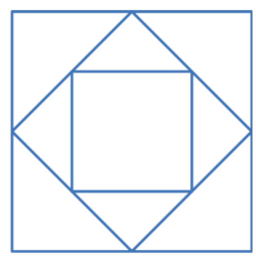
$$3, -1, \frac{1}{3}, -\frac{1}{9}, \ldots$$

$$0.3, 0.06, 0.012, \ldots$$

We know that these sequences represent the geometric progressions (GP) of the type a, ar, ar^2, ar^3, … where both a and r are non-zero numbers, a is the first term and r is the common ratio.

Recollect that, if the terms are denoted as $t_1, t_2, t_3, \ldots, t_n$ in the GP, then the n^{th} term is given by $t_n = ar^{n-1}$.

6.2 Sum of the first *n* terms of a Geometric Progression



A model of a sculpture is required to be made by using square sheets of cardboard. The first square has a side of 40 cm. The midpoints of its sides are joined to form another square and a cardboard of same size is pasted on the original square. Similarly, a smaller square is pasted on the second square. This is repeated till the 10^{th} square is pasted.

How can we find the total area of cardboard required for this model?

We notice that the sides of the squares will form a GP $40, 20\sqrt{2}, 20, \ldots$ and the areas will form the GP $1600, 800, 400, \ldots$

Let us explore if we can find the sum of the first 10 terms of this sequence.

The **sum of the first *n* terms of a GP (S_n)** can be easily obtained.

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \quad \ldots (1)$$

Multiplying both sides of equation (1) by r

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n \quad \ldots (2)$$

Subtracting equation (2) from (1) we get

$$(1-r)S_n = a - ar^n$$

$$S_n = \frac{a(1-r^n)}{1-r}, \; r \neq 1$$

So, the area of all the 10 squares, whose GP is 1600, 800, 400, … can be calculated by using the above result.

Hence the area is

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{1600 \times \left[1-\left(\frac{1}{2}\right)^{10}\right]}{1-\left(\frac{1}{2}\right)}$$

$$= 3200\left[1-\left(\frac{1}{2}\right)^{10}\right]$$

$$= 3200\left[\frac{1023}{1024}\right]$$

$$= \frac{25 \times 1023}{8} = 3196.87 \text{ sq. cm.}$$

So, the total area of cardboard required for this model is 3196.87 sq cm.

Let us take some more examples.

Example 1: Find the sum upto n terms of the sequence $3, -1, \frac{1}{3}, -\frac{1}{9}, \ldots$

Solution: Here $\frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{-1}{3} = r$, therefore it is a GP,

$a = 3$ and $r = \frac{-1}{3} \neq 1$

So $$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{3 \times \left[1-\left(\frac{-1}{3}\right)^n\right]}{1-\left(\frac{-1}{3}\right)}$$

$$= \frac{9}{4}\left[1-\left(\frac{-1}{3}\right)^n\right]$$

Example 2: Find the sum upto n terms of the sequence 0.3, 0.06, 0.012, …

Solution: Here $\frac{t_2}{t_1} = \frac{t_3}{t_2} = 0.2 = r$, therefore it is a GP.

$a = 0.3$ and $r = 0.2 \neq 1$

So
$$S_n = \frac{a(1-r^n)}{1-r}$$
$$= \frac{(0.3)\left[1-(0.2)^n\right]}{1-(0.2)}$$
$$= \frac{3}{8}\left[1-(0.2)^n\right]$$

Example 3: Find the sum upto n terms of the series 7 + 77 + 777 + …

Solution: The given series can be written as

$$S_n = 7(1 + 11 + 111 + \dots \text{ up to } n \text{ terms})$$
$$= \frac{7}{9}(9 + 99 + 999 + \cdots \text{ up to } n \text{ terms})$$
$$= \frac{7}{9}[(10-1) + (100-1) + (1000-1) + \cdots \text{ up to } n \text{ terms }]$$
$$= \frac{7}{9}[(10+100+1000+\dots \text{ to } n \text{ terms}) - n]$$
$$= \frac{7}{9}\left[\frac{10\,[(10)^n - 1]}{9} - n\right]$$

Therefore
$$S_n = \frac{7}{9}\left[\frac{10}{9}[(10)^n - 1] - n\right]$$

6.3 Infinite Geometric Progression

A boy observed an insect, which was moving along a straight path.

He saw that it was able to move 2 cm in the first minute and thereafter it is covering 1 cm, 0.5 cm, 0.25 cm in the second, third and subsequent minutes and carries on and on.

The boy notices the sequence of distances covered by the insect form a GP as 2, 1, 0.5, … .

He then, marked a point 5 cm away from the starting point.

Will this insect be able to reach this mark or not? He wondered.

Let us see if the concept of a GP will help the boy or not.

Can you guess the number of terms this GP will have?

What if it has infinite terms?

That brings us to explore, if we can find the sum of infinite terms of a GP or not.

In the formula of S_n for a GP, above we notice the expression r^n.

If x is a real number, then

$x > x^2 > x^3 > x^4 > x^5 > x^6$.

Is this ever possible?

When you think about this question, try making two cases for x.

Case 1: When $|x| < 1$ which means $-1 < x < 1$

Case 2: When $|x| > 1$ which means $x > 1$ or $x < -1$

It is clearly seen that $r > r^2 > r^3 > r^4 > r^5 > r^6$. is true only in the first case.

Let us now refer back to our GP and the formula for its sum to n terms

$$S_n = \frac{a(1-r^n)}{1-r}$$

Observe if $|r| < 1$ which means $-1 < r < 1$ then $r > r^2 > r^3 > r^4 > r^5 > r^6$.

So, the value of r^n decreases as n is increased and it will approach to zero as n approaches to higher values.

We say that r^n will approach to zero as n approaches to infinity. This enables us to find the sum to infinite terms of a GP, when $-1 < r < 1$.

$$\mathbf{S_\infty = \frac{a}{1-r} \text{ only if } -1 < r < 1}$$

Coming back to the boy and insect, the boy notices the sequence of distances covered by the insect form a GP as 2, 1, 0.5, … where $a = 2$ and $r = \frac{1}{2}$.

Since $\frac{1}{2} < 1$, he could find $S_\infty = \frac{a}{1-r} = \frac{2}{1-\frac{1}{2}} = 4$.

This meant that the insect could not reach that mark ever.

Early Egyptian and Babylonian tablets show the examples of infinite GP. Sumerian tablets (c. 2000 BC) show a GP with a base of 3 and multiplier $\frac{1}{2}$.

Mādhava of the Kerala School of mathematics stated the value of π in terms of infinite series and found it as 3.141592653592...which is correct up to 11 decimal places.

He used various geometrical proofs for the convergence of finite or infinite geometric series. This was quoted by *Śaṅkara Vāriyar* in his commentary *Yukti-dīpikā* and *Kriyā-kramakarī*.

Example 4: Find the sum to infinite terms of the GP.

$$\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\ldots$$

Solution: Here $a = \frac{1}{2}$ and $r = \frac{1}{4}$ and $\left|\frac{1}{4}\right|<1$

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$$

Example 5: An infinite GP has the first term 'x' and sum 4. What are all the possible values that x can take?

Solution: Let the common ratio of the GP be r. Then

$$S_\infty = \frac{x}{1-r} = 4$$

Therefore $\frac{x}{4} = 1-r$ or $r = 1-\frac{x}{4}$

Since $-1 < r < 1$, we get $-1<1-\frac{x}{4}<1$

Adding or subtracting the same number in both the sides of an inequality does not change the sign.

Subtracting 1 from all the three, we get $-2<-\frac{x}{4}<0$

We know that multiplication by a negative number changes the sign of an inequality.

Multiplying by –4, we get $8 > x > 0$.

This means that **x can be any real number between 0 and 8**.

Example 6: The sum of an infinite series of GP is 3 and sum of the squares of its terms is $\frac{9}{2}$. Find the sum of the cubes of the terms of the original GP.

Solution: Let the series be

$$S_\infty = a + ar + ar^2 + ar^3 + \ldots$$

Then $$S_\infty = \frac{a}{1-r} = 3 \quad \ldots (1)$$

The series $a^2 + a^2r^2 + a^2r^4 + a^2r^6 + \ldots$ is also a GP with first term as a^2 and common ratio r^2.

Therefore $$\frac{a^2}{1-r^2} = \frac{9}{2} \quad \ldots (2)$$

Squaring both sides of equation (1) and dividing by equation (2) respectively.

$$\frac{a^2}{(1-r)^2} \times \frac{1-r^2}{a^2} = 9 \times \frac{2}{9}$$

Solving we get $\frac{1+r}{1-r} = 2$ and therefore $r = \frac{1}{3}$.

Substituting this value in equation (1) we get $a = 2$.

Therefore $a^3 = 8$ and $r^3 = \frac{1}{27}$

The sum to infinite terms of the cubes

$$= a^3 + a^3r^3 + a^3r^6 + a^3r^{18} + \ldots = \frac{8}{1-\frac{1}{27}} = \frac{108}{13}$$

EXERCISE 6.1

1. Find the sum of the series 0.15 + 0.015 + 0.0015 + … to 15 terms.
2. Find the sum to n terms of the series 0.9 + 0.99 + 0.999 + …

   [Hint: write(0.9 = 1 – 0.1) and (0.99 = 1 – 0.01) and so on…]
3. Find the sum to n terms of the series 5 + 55 + 555 + …
4. The sum of the first n terms of the sequence 3, 6, 12, … is 381. Find n.

5. Find the sum to n terms of the series

$$x(x+y)+x^2(x^2+y^2)+x^3(x^3+y^3)+\ldots$$

6. Find the sum of the infinite terms of the series

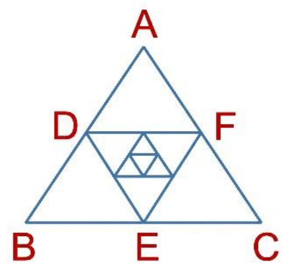
$$1+\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^2}+\frac{1}{3^4}+\frac{1}{2^3}+\frac{1}{3^6}+\ldots$$

7. The sum of an infinite series of GP is 6 and sum of the squares of these terms is 12. Find the common ratio of the original GP.

8. If the sum to infinity of the series $1+r+r^2+r^3+\ldots$ is S, given that $|r|<1$ then write r in terms of S.

9. Find the value of $4^{\frac{1}{2}} \cdot 4^{\frac{1}{4}} \cdot 4^{\frac{1}{8}} \cdot 4^{\frac{1}{16}} \ldots$ to ∞.

10. The midpoints D, E and F of the sides of an equilateral triangle ΔABC are joined to form another smaller equilateral triangle. This process is repeated in the ΔDEF and so on indefinitely, getting smaller and smaller triangles. If each side of the ΔABC is 16 cm, find the sum of the perimeters of all the triangles, so formed.



11. Let $f(x) = 2x + 1$, then find the number of values of x for which $f(x)$, $f(2x)$, $f(4x)$ are in a GP.

12. If $t_1, t_2, t_3, t_4, \ldots$ are the terms of a GP whose common ratio is r such that $-1 < r < 1$, then evaluate the following:

$$\frac{t_1-t_3+t_5-\ldots}{t_2-t_4+t_6-\ldots}$$

13. A particular ball rebounds $\left(\frac{3}{5}\right)^{\text{th}}$ of the height from which it falls, whenever it strikes the floor. Find the total distance it covers before coming to rest, if it falls due to gravity from a height of 90 meters.

6.4 Method of Differences and Combinatorics

Let us now observe a few progressions in which the differences of the successive terms are in an AP.

The most common sequences of these type are the square numbers and the triangular numbers.

Square Numbers: 1, 4, 9, 16, 25, …

Triangular Numbers: 1, 3, 6, 10, 15, …

Example 7: Find the nth term of the sequence 3, 7, 13, 21, 31, … and hence find its 8th term.

Solution: Let us check the differences of consecutive terms.

$t_2 - t_1 = 4, \quad t_3 - t_2 = 6, \quad t_4 - t_3 = 8$ and so on…

We notice that the differences of the subsequent terms from their immediately previous terms form an AP.

Let S be the sum of the first n terms of this series and let t_n be its n^{th} term.

$$S = 3 + 7 + 13 + 21 + 31 + ... + t_n \qquad ... (1)$$

Also $$S = \quad 3 + 7 + 13 + 21 + 31 + ... + t_n \qquad ... (2)$$

Subtracting both sides of the equation (2) from the corresponding sides of (1)

$$t_n = 3 + [4 + 6 + 8 + \ldots + (t_n - t_{n-1})] \qquad ... (3)$$

Clearly $4 + 6 + 8 + \ldots + (t_n - t_{n-1})$ is an AP with first term as 4 and common difference as 2 and number of terms is (n – 1).

Therefore $$t_n = 3 + \frac{(n-1)}{2}[8 + (n-2)2] \qquad \text{using the } S_n \text{ of an AP.}$$

$$= 3 + (n - 1)(n + 2)$$

$$= n^2 + n + 1$$

and $$t_8 = 8^2 + 8 + 1 = 73$$

6.4.1 Alternative Shorter Method for t_n

Let us write the given sequence followed by the subsequent differences in rows.

3 7 13 21 31 … (Given sequence) Row1

4 6 8 10 … (The differences) Row2

2 2 2 … (The differences of the differences) Row3

Let

b = first term of the original sequence

a = first term of the 1st differences

d = first term of the 2nd differences

d_1 = first term of the 3rd differences

And so on…

Then the n^{th} term is given by

$$t_n = b \cdot C(n-1, 0) + a \cdot C(n-1, 1) + d \cdot C(n-1, 2)$$
$$= b + a(n-1) + d\frac{(n-1)(n-2)}{2}$$

Recollect $C(n, r) = \frac{n!}{r!(n-r)!}$ is the number of combinations of r objects out of n.

Using the above formula, we notice that

$C(n-1, 0) = 1$ and $C(n-1, 1) = n-1$

$C(n-1, 2) = \frac{(n-1)(n-2)}{2}$ and $C(n-1, 3) = \frac{(n-1)(n-2)(n-3)}{3 \times 2}$ and so on.

Therefore, in the above progression $b = 3$, $a = 4$ and $d = 2$

$$t_n = 3 + 4(n-1) + 2\frac{(n-1)(n-2)}{2}$$
$$t_n = n^2 + n + 1$$

and $$t_8 = 8^2 + 8 + 1 = 73$$

6.4.2 Summation to n terms

By using combinatorics to this type of a sequence we can also find the sum to the first n terms using the formula below:

$$S_n = b \cdot C(n, 1) + a \cdot C(n, 2) + d \cdot C(n, 3)$$
$$= bn + a\frac{(n)(n-1)}{2} + d\frac{n(n-1)(n-2)}{3 \times 2}$$

The concept of Methods of Differences in Progressions can be found as *Vārasaṅkalita* discussed by *Nārāyaṇa* in *Gaṇita Kaumudī* and also found in *Sama-ghāta-saṅkalita* in *Yukti-bhāṣa* after Bhāskara's time.

Vārasaṅkalita is defined as follows:

1 + 2 + 3 + … up to n terms

1 + 3 + 6 + 10 + … up to n terms

1 + 4 + 10 + 20 + … up to n terms

The last two series are the ones in which subsequent differences of consecutive terms form an AP or reducible to AP.

Example 8: Find the sum to the first n terms of the sequence mentioned in example 7. Also find the sum of the first 8 terms using the above formula.

Solution: The sequence is 3, 7, 13, 21, 31, … in which $b = 3$, $a = 4$ and $d = 2$

$$s_n = 3n + 4\,\frac{n(n-1)}{2} + 2\,\frac{n\,(n-1)(n-2)}{3\times 2} = \frac{n}{3}(n^2 + 3n + 5)$$

$$s_8 = \frac{8}{3}(8^2 + 3\times 8 + 5) = \frac{744}{3} = 248$$

Example 9: Find the 10th term and the sum to 10 terms of the sequence 2, 5, 9, 18, 36,…

Solution: Here $n = 10$. The given sequence can be written as

2	5	9	18	36	…
3	4	9	18		…
1	5	9			…
4	4				…

If we name the first element in each row as b, a, d and d_1 respectively, we get $b = 2$, $a = 3$, $d = 1$ and $d_1 = 4$

The n^{th} term is given by

$$t_n = b\cdot C(n-1,\ 0) + a\cdot C(n-1,\ 1) + d\cdot C(n-1, 2) + d_1\cdot C(n-1,\ 3)$$

$$t_{10} = b\cdot C(9,\ 0) + a\cdot C(9,\ 1) + d\cdot C(9, 2) + d_1\cdot C(9, 3)$$

$$t_{10} = 2\cdot C(9,\ 0) + 3\cdot C(9,\ 1) + 1\cdot C(9,\ 2) + 4\cdot C(9,\ 3)$$

$$= 2 + 3(9) + \frac{(9)(8)}{2} + 4\frac{(9)(8)(7)}{3\times 2} = 401$$

The sum to the first 10 terms is

$$S_n = b\cdot C(n,\ 1) + a\cdot C(n,\ 2) + d\cdot C(n, 3) + d_1\cdot C(n,\ 4)$$

$$S_{10} = b\cdot C(10,\ 1) + a\cdot C(10,\ 2) + d\cdot C(10, 3) + d_1\cdot C(10, 4)$$

$$S_{10} = 2\cdot C(10,\ 1) + 3\cdot C(10,\ 2) + 1\cdot C(10,\ 3) + 4\cdot C(10,\ 4)$$

$$= 2(10) + 3\frac{(10)(9)}{2} + \frac{(10)(9)(8)}{3\times 2} + 4\frac{(10)(9)(8)(7)}{4\times 3\times 2}$$

$$= 20 + 135 + 120 + 840 = 1115$$

EXERCISE 6.2

1. Find the n^{th} term and the sum of the first n terms of the series

$$1 + 9 + 24 + 46 + 75 + ...$$

2. Find the 10^{th} term and the sum of the first 10 terms of the series

$$4 + 5 + 9 + 16 + 26 + \cdots$$

3. Find the n^{th} term and the sum of the first 12 terms of the series

$$3 + 6 + 11 + 18 + 27 + \cdots$$

4. Find the 8^{th} term and the sum of the first 8 terms of the series

$$4 + 13 + 28 + 49 + 76 + \cdots$$

Summary

1. The sum of the first n terms of a GP is given by $S_n = \frac{a(1-r^n)}{1-r}$, $r \neq 1$, where a and r are the first term and common ratio of the GP.

2. The sum to infinite terms of a GP is given by $S_\infty = \frac{a}{1-r}$ only if $-1 < r < 1$.

3. The n^{th} term of a series in which the subsequent differences form an AP is given by

$$t_n = b \cdot C(n-1, 0) + a \cdot C(n-1, 1) + d \cdot C(n-1, 2)$$
$$= b + a(n-1) + d\frac{(n-1)(n-2)}{2}$$

where b is the first term of the given series and a and d are the first term and common difference respectively of the AP.

4. The sum of the first n term of a series in which the subsequent differences form an AP is given by

$$S_n = b \cdot C(n, 1) + a \cdot C(n, 2) + d \cdot C(n, 3)$$
$$= bn + a\frac{(n)(n-1)}{2} + d\frac{n(n-1)(n-2)}{3 \times 2}$$